TD Algèbre

Rappel :
On dit que $(E, \diamond)$ est un groupe si :

0) $(E, \diamond)$ est un magma
1) $\diamond$ est associative : $\forall u, v, w \in E \; (u \diamond v) \diamond w = u \diamond (v \diamond w)$
2) $\diamond$ a un élément neutre : $\exists e \in E, \forall u \in E, u \diamond e = e \diamond u = u$
3) Symétri~~que~~ $\forall u \in G \;\; \exists u' \in G, \; u \diamond u' = u' \diamond u = e$

4) Commutative $\forall u, v \in E, \; u \diamond v = v \diamond u$

Exercice 1 :

0) $G =]-1, 1[$

$$\Delta : G \times G \longrightarrow G ? \quad \text{on teste en remplaçant } x \text{ et } y \text{ par }]-1, 1[$$
$$(x, y) \longmapsto \frac{x+y}{1+x \cdot y}$$

On suppose
$-1 < x < 1$
$-1 < y < 1$

On veut montrer

$$-1 < \frac{x+y}{1+x \cdot y} < 1 \iff \left|\frac{x+y}{1+x \cdot y}\right| < 1 \iff \left(\frac{x+y}{1+xy}\right)^2 < 1$$

$$\iff \frac{(x+y)^2}{(1+xy)^2} < 1$$

$$\iff (x+y)^2 < (1+xy)^2$$

$$\iff 0 < (1+xy)^2 - (x+y)^2$$

$$\iff 0 < 1 + 2xy + x^2y^2 - (x^2 + 2xy + y^2)$$

$$\iff 0 < 1 + 2xy + x^2y^2 - x^2 - 2xy - y^2 = x^2y^2 - x^2 + 1 - y^2$$
$$= -x^2(1-y^2) + 1 - y^2 = \underbrace{(1-x^2)}_{>0}\underbrace{(1-y^2)}_{>0}$$

1) $x \Delta y = \dfrac{x+y}{1+xy}$

$\forall (x, y, z) \in G^3$
$\forall x, y, z \in G$

$$(x \Delta y) \Delta z = \left(\frac{x+y}{1+(x \cdot y)}\right) \Delta z = \frac{\frac{x+y}{1+xy} + z}{1 + \left(\frac{x+y}{1+xy}\right) z} \times \frac{1+xy}{1+xy} = \frac{\left(\frac{x+y}{1+xy} + z\right)(1+xy)}{\left[1 + \left(\frac{x+y}{1+xy}\right) z\right]\left[1+xy\right]}$$

$$= \frac{x+y+z+xyz}{1+xy+yz+zx}$$

$$x \Delta (y \Delta z) = x \Delta \frac{y+z}{1+yz} = \frac{x + \frac{y+z}{1+y \cdot z}}{1 + x \, \frac{y+z}{1+y \cdot z}} \times \frac{1+yz}{1+yz} = \frac{y+z+x+xyz}{1+yz+xy+xz}$$

$\forall x, y \in G$, $x \Delta y = \frac{x+y}{1+xy}$

on cherche $e \in G$ tels que pour tous x de G

$e \Delta x = x$ et $x \Delta e = x$

$e \Delta x = \frac{e+x}{1+e \cdot x} = x$

$\frac{e+x}{1+e \cdot x} = x$

$e+x = x(1+ex)$

$e+x = x+ex^2$

$0 = e(1-x^2)$

> 0

$e = 0 \in G$ $\left.\begin{array}{l} \frac{0+x}{1+0 \cdot x} = x \\ \frac{x+0}{1+x \cdot 0} = x \end{array}\right\} \forall x \in G$

3) Symétrie

$\forall x \in G$

$\exists x' \in G$ $\boxed{x' = -x \in G}$

$x \Delta x' = e = 0$

$\frac{x+x'}{1+xx'} =$

$5 + ? = 0$

$5 \times ? = 1$

$\forall x, y \in G$

$x \Delta y = \frac{x+y}{1+xy} = \frac{y+x}{1+y \cdot x} = y \Delta x$

Ex 2] $f_{a,b} : \mathbb{R} \mapsto \mathbb{R}$
$x \mapsto ax+b$

$F = \{f_{a,b}, a \in \mathbb{R}^*, b \in \mathbb{R}\}$

muni de la composition $\circ$

Rappel : $f \circ g : \mathbb{R} \mapsto \mathbb{R}$
$x \mapsto f(g(x))$

$(f \circ g)(x) = f(g(x))$

0) $(F, \circ)$ est-elle un magma

$\forall f_{a,b}$ et $f_{\alpha,\beta} \in F$

$\forall x \in \mathbb{R}$

$f_{a,b} \circ f_{\alpha,\beta}(x) = f_{a,b}(f_{\alpha,\beta}(x)) = f_{a,b}(\alpha x + \beta) = a(\alpha x + \beta) + b = (a\alpha)x + a\beta + b =$

$f_{a\alpha, a\beta+b}(x)$

En résumé $\boxed{f_{a,b} \circ f_{\alpha,\beta} = f_{a\alpha, a\beta+b}} \in F$

② Elt neutre

$\exists e = f_{a_e, b_e} \in F$, $\forall f_{a,b} \in F$

$f_{a,b} \circ f_{a_e, b_e} = f_{a,b}$?

$f_{a a_e, a b_e + b} = f_{a,b}$

$\begin{cases} a a_e = a \\ a b_e + b = b \end{cases} \Leftrightarrow \begin{cases} a_e = 1 \\ b_e = 0 \end{cases}$

$f_{1,0}$ est l'élément neutre

$f_{a,b} \circ f_{1,0} = f_{a \times 1, a \times 0 + b} = f_{a,b}$

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1) Montrons l'associativité de la loi $\circ$.

$f_{a,b} \circ f_{a',b'} = f_{aa',ab'+b}$

$E, \diamond$
1) Associativité
$\forall u, v, w \in E$
$(u \diamond v) \diamond w = u \diamond (v \diamond w)$

$(f_{a,b} \circ f_{a',b'}) \circ f_{a'',b''} = f_{aa',ab'+b} \circ f_{a'',b''} = f_{aa'a'', aa' \times b'' + ab' + b}$

$f_{a,b} \circ (f_{a',b'} \circ f_{a'',b''}) = f_{a,b} \circ (f_{a'a'', a'b''+b'}) = f_{aa'a'', a(a'b''+b')+b} = f_{aa'a'', aa' \times b'' + ab' + b}$

$\circ$ est associative

③ Symetrique

On à trouver que $e = \underline{f_{1,0}}$

$(E, \diamond)$
3) $\forall u \in E, \exists u' \in E$
tels que
$u \diamond u' = u' \diamond u = e$

$f_{a,b} \circ f_{\alpha,\beta} = f_{1,0}$

But Trouver α et β (en fonction de a et b)

$f_{a,b} \circ f_{\alpha,\beta} = f_{a\alpha, a\beta+b} = f_{1,0}$

$$\Leftrightarrow \begin{cases} a\alpha = 1 \\ a\beta + b = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha = \frac{1}{a} \\ \beta = -\frac{b}{a} \end{cases}$$

$f_{1/a, -\frac{b}{a}}$ est le symétrique de $f_{a,b}$

et $f_{1/a, -b/a} \circ f_{a,b} = f_{\frac{1}{a} \times a, \frac{1}{a} \times b - \frac{b}{a}} = f_{1,0}$

Pour savoir si commutative on compare les deux expressions.
en changeant de sens.

$f_{a,b} \circ f_{a',b'} = f_{aa', ab'+b}$

$f_{a',b'} \circ f_{a,b} = f_{a'a, a'b+b'}$

② $m_{a,b} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$

$M = \{ m_{a,b} , a \in \mathbb{R}^*, b \in \mathbb{R} \}$

$(M, \cdot)$
produit matriciel

$m_{a,b} \cdot m_{a',b'} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a' & b' \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} aa' & ab'+b \\ 0 & 1 \end{bmatrix} = m_{aa', ab'+b}$

③ $(F, \circ) \underset{\text{isomorphe}}{\simeq} (M, \cdot)$

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