

Rappel:

On dit que (E, \diamond) est un groupe si:

0) (E, \diamond) est un magma

1) \diamond est associative: $\forall u, v, w \in E: (u \diamond v) \diamond w = u \diamond (v \diamond w)$

2) \diamond a un élément neutre: $\exists e \in E, \forall u \in E, u \diamond e = e \diamond u = u$

3) Symétrisable $\forall u \in E \exists u' \in E, u \diamond u' = u' \diamond u = e$

4) Commutative $\forall u, v \in E, u \diamond v = v \diamond u$

Exercice 1:

0) $G =]-1, 1[$

$\Delta: G \times G \mapsto G$ on teste en remplaçant x et y par $]-1, 1[$

$(x, y) \mapsto \frac{x+y}{1+x \cdot y}$

On suppose

$-1 < x < 1$

$-1 < y < 1$

On veut montrer

$$-1 < \frac{x+y}{1+x \cdot y} < 1 \Leftrightarrow \left| \frac{x+y}{1+x \cdot y} \right| < 1 \Leftrightarrow \left(\frac{x+y}{1+x \cdot y} \right)^2 < 1$$

$$\Leftrightarrow \frac{(x+y)^2}{(1+x \cdot y)^2} < 1$$

$$\Leftrightarrow (x+y)^2 < (1+x \cdot y)^2$$

$$\Leftrightarrow 0 < (1+x \cdot y)^2 - (x+y)^2$$

$$\Leftrightarrow 0 < 1 + 2x \cdot y + x^2 \cdot y^2 - (x^2 + 2x \cdot y + y^2)$$

$$\Leftrightarrow 0 < 1 + 2x \cdot y + x^2 \cdot y^2 - x^2 - 2x \cdot y - y^2 = x^2 \cdot y^2 - x^2 + 1 - y^2$$

$$= x^2(1 - y^2) + 1 - y^2 = (1 - x^2)(1 - y^2)$$

$\begin{matrix} > 0 & & > 0 \end{matrix}$

1) $x \Delta y = \frac{x+y}{1+x \cdot y}$

$\forall (x, y, z) \in G^3$
 $\forall x, y, z \in G$

$$(x \Delta y) \Delta z = \left(\frac{x+y}{1+x \cdot y} \right) \Delta z = \frac{\frac{x+y}{1+x \cdot y} + z}{1 + \left(\frac{x+y}{1+x \cdot y} \right) \cdot z} \times \frac{1+x \cdot y}{1+x \cdot y} = \frac{\left(\frac{x+y}{1+x \cdot y} + z \right) (1+x \cdot y)}{\left[1 + \left(\frac{x+y}{1+x \cdot y} \right) \cdot z \right] [1+x \cdot y]}$$

$$= \frac{x \cdot y + z + x \cdot y \cdot z}{1 + x \cdot y + y \cdot z + z \cdot x}$$

$$x \Delta (y \Delta z) = x \Delta \frac{y+z}{1+y \cdot z} = \frac{x + \frac{y+z}{1+y \cdot z}}{1 + x \cdot \frac{y+z}{1+y \cdot z}} = \frac{x(1+y \cdot z) + y+z}{1+y \cdot z + x \cdot y + x \cdot z} = \frac{x + y \cdot z + x \cdot y + z}{1 + x \cdot y + y \cdot z + z \cdot x}$$

$$\forall x, y \in G, x \Delta y = \frac{x+y}{1+xy}$$

on cherche $e \in G$ tels que pour tous x de G

$$\begin{aligned} e \Delta x &= x & e \Delta x &= \frac{e+x}{1+e \cdot x} = x \\ x \Delta e &= x \end{aligned}$$

$$\begin{aligned} e+x &= x \\ 1+e \cdot x & \end{aligned}$$

$$\begin{aligned} e \cdot x &= x(1+e \cdot x) \\ e+x &= x+e \cdot x^2 \\ 0 &= e(1-x^2) \\ &> 0 \end{aligned}$$

$$e = 0 \in G \quad \left. \begin{aligned} \frac{0+x}{1+0 \cdot x} &= x \\ \frac{x+0}{1+x \cdot 0} &= x \end{aligned} \right\} \forall x \in G$$

3) Symétrie

$$\forall x \in G$$

$$\exists x' \in G \quad \boxed{x' = -x \in G}$$

$$x \Delta x' = e = 0$$

$$\frac{x+x'}{1+x \cdot x'} =$$

$$\begin{aligned} 5+8 &= 0 \\ 5x &= 1 \end{aligned}$$

$$\forall x, y \in G$$

$$x \Delta y = \frac{x+y}{1+xy} = \frac{y+x}{1+yx} = y \Delta x$$

$$\text{Ex 2)} f, a, b: \mathbb{R} \rightarrow \mathbb{R} \\ x \mapsto a \cdot x + b$$

$$F = \{ f, a, b, a \in \mathbb{R}^*, b \in \mathbb{R} \}$$

muni de la composition \circ

$$\text{Rappel: } f \circ g: \mathbb{R} \rightarrow \mathbb{R} \\ x \mapsto f(g(x))$$

1) (F, \circ) est-elle un magma?

$$\forall f, a, b \text{ et } f', a', b' \in F \\ \forall x \in \mathbb{R}$$

$$(f \circ g)(x) = f(g(x))$$

$$f, a, b \circ f', a', b'(x) = f, a, b(f', a', b'(x)) = f, a, b(a'x + b') = a(a'x + b') + b = f, a \cdot a', a \cdot b' + b = f, a \cdot a', a \cdot b' + b(x)$$

$$\text{En résumé } \boxed{f, a, b \circ f', a', b' = f, a \cdot a', a \cdot b' + b} \in F$$

2) Élément neutre

$$\exists e = f, a, b \in F, \forall f, a, b \in F \\ f, a, b \circ f, a, b = f, a, b \\ f, a, b \circ f, a, b = f, a, b$$

$f, 1, 0$ est l'élément neutre

$$\begin{aligned} f, a, b \circ f, 1, 0 &= f, a \cdot 1 + a \cdot 0 + b = f, a, b \\ f, 1, 0 \circ f, a, b &= f, 1 \cdot a + 1 \cdot b + 0 = f, a, b \end{aligned}$$

$$f, a \cdot a = a \Leftrightarrow \begin{cases} a \cdot a = 1 \\ a \cdot b = 0 \end{cases}$$

1) Montrer l'associativité de la loi \circ .

$$f(a, b) \circ f(a', b') = f(aa', ab' + b)$$

E, \emptyset

1) Associativité

$\forall u, v, w \in E$

$$(u \circ v) \circ w = u \circ (v \circ w)$$

$$(f(a, b) \circ f(a', b')) \circ f(a'', b'') = f(aa', ab' + b) \circ f(a'', b'') = f(aa'a'', aa'x b'' + ab' + b)$$

$$f(a, b) \circ (f(a', b') \circ f(a'', b'')) = f(a, b) \circ (f(a'a'', a'b'' + b')) = f(aa'a'', a(ab'b'' + b') + b) = f(aa'a'', aa'x b'' + ab' + b)$$

\circ est associative

③ Symétrique

On a trouver que $e = \underline{f(1, 0)}$.

$$f(a, b) \circ f(x, y) = f(1, 0)$$

Bes Trouver x et y (en fonction de a et b)

$$f(a, b) \circ f(x, y) = f(ax, ay + b) = f(1, 0)$$

$$\Leftrightarrow \begin{cases} ax = 1 \\ ay + b = 0 \end{cases} \Leftrightarrow \begin{cases} x = \frac{1}{a} \\ y = -\frac{b}{a} \end{cases}$$

$f(1/a, -b/a)$ est le symétrique de $f(a, b)$

$$f(1/a, -b/a) \circ f(a, b) = f(\frac{1}{a} \times a, \frac{1}{a} \times b - \frac{b}{a}) = f(1, 0)$$

Pour savoir si commutative on compare les deux expressions en changeant de sens.

$$f(a, b) \circ f(a', b') = f(aa', ab' + b)$$

$$f(a', b') \circ f(a, b) = f(a'a, a'b + b')$$

$$② m(a, b) = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

$$M = \{m(a, b), a \in \mathbb{R}^*, b \in \mathbb{R}\}$$

(M, \circ) produit matriciel

$$m(a, b) \circ m(a', b') = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a' & b' \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} aa' & ab' + b \\ 0 & 1 \end{bmatrix} = m_{aa', ab' + b}$$

③ $(F, \circ) \cong (H, +)$
isomorphe

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1) Montrons l'associativité de la loi \circ .

$$f(a, b \circ f(a', b')) = f(a, b' + b) = f(aa', ab' + b)$$

E, ϕ

1) Associativité

$$\forall u, v, w \in E$$

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$$f(a, b \circ (f(a', b') \circ f(a'', b''))) = f(a, b \circ (f(a'a'', a'b'' + b''))) = f(aa'a'', a(a'b'' + b') + b) = f(aa'a'', aa'x b'' + ab' + b)$$

\circ est associative

③ Symétrique

On a trouver que $e = f(1, 0)$.

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Bien Trouver x et y (en fonction de a et b)

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$$f(a, b \circ f(a', b')) = f(a, b' + b) = f(aa', ab' + b)$$

$$f(a', b' \circ f(a, b)) = f(a'a, a'b + b')$$

$$\textcircled{2} m_{a, b} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

$$M = \{m_{a, b}, a \in \mathbb{R}^*, b \in \mathbb{R}\}$$

(M, \circ) produit matriciel

$$m_{a, b} \circ m_{a', b'} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a' & b' \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} aa' & ab' + b \\ 0 & 1 \end{bmatrix} = m_{aa', ab' + b}$$

③ $(F, \circ) \simeq (M, \circ)$
isomorphe

$$\forall x, y \in G, x \Delta y = \frac{x+y}{1+xy}$$

on cherche $e \in G$ tels que pour tous x de G

$$e \Delta x = x$$

$$e \Delta x = \frac{e+x}{1+e \cdot x} = x$$

$$e+x = x$$

$$\forall e \cdot x$$

$$e \cdot x = x(1+e \cdot x)$$

$$e+x = x+e \cdot x^2$$

$$0 = e(1-x^2)$$

$$> 0$$

$$e = 0 \in G$$

$$\frac{0+x}{1+0 \cdot x} = x$$

$$\frac{x+0}{1+x \cdot 0} = x$$

$$\forall x \in G$$

3) Symétrie

$$\forall x \in G$$

$$5 + \frac{0}{5} = 0$$

$$5x = 1$$

$$\exists x' \in G$$

$$x' = -x \in G$$

$$x \Delta x' = e = 0$$

$$\forall x, y \in G$$

$$x \Delta y = \frac{x+y}{1+xy} = \frac{y+x}{1+yx} = y \Delta x$$

$$\frac{x+x'}{1+xx'} =$$

$$1+xx'$$

$$\text{Ex 2) } f: a, b: \mathbb{R} \mapsto \mathbb{R}$$

$$x \mapsto a \cdot x + b$$

$$F = \{ f_{a,b}, a \in \mathbb{R}^*, b \in \mathbb{R} \}$$

muni de la composition \circ

Rappel: $f \circ g: \mathbb{R} \mapsto \mathbb{R}$

$$x \mapsto f(g(x))$$

1) (F, \circ) est-elle un magma?

$$(f \circ g)(x) = f(g(x))$$

$$\forall f_{a,b} \text{ et } f_{\alpha,\beta} \in F$$

$$\forall x \in \mathbb{R}$$

$$f_{a,b} \circ f_{\alpha,\beta}(x) = f_{a,b}(f_{\alpha,\beta}(x)) = f_{a,b}(\alpha x + \beta) = f_{a,b}(\alpha x + \beta) = a(\alpha x + \beta) + b = (a\alpha)x + a\beta + b = f_{a\alpha, a\beta+b}(x)$$

$$f_{a\alpha, a\beta+b}(x)$$

$$\text{En résumé } f_{a,b} \circ f_{\alpha,\beta} = f_{a\alpha, a\beta+b} \in F$$

2) Est neutre

$f_{1,0}$ est l'élément neutre

$$\exists e = f_{a,b}, b \in F, \forall f_{\alpha,\beta} \in F$$

$$f_{\alpha,\beta} \circ f_{a,b} = f_{\alpha,\beta} \circ f_{a,b} = f_{\alpha,\beta}$$

$$f_{\alpha,\beta} \circ f_{a,b} = f_{\alpha,\beta} \circ f_{a,b} = f_{\alpha,\beta}$$

$$f_{a\alpha, a\beta+b} = f_{\alpha,\beta} \Leftrightarrow \begin{cases} a\alpha = \alpha \\ a\beta+b = \beta \end{cases}$$

$$f_{a,b} \circ f_{1,0} = f_{a \cdot 1, a \cdot 0 + b} = f_{a,b}$$

$$f_{1,0} \circ f_{a,b} = f_{1 \cdot a, 1 \cdot b + 0} = f_{a,b}$$