TD Algèbre

Rappel :
On dit que $(E, \diamond)$ est un groupe si :

0) $(E, \diamond)$ est un magma
1) $\diamond$ est associative : $\forall u, v, w \in E \quad (u \diamond v) \diamond w = u \diamond (v \diamond w)$
2) $\diamond$ a un élément neutre : $\exists e \in E, \forall u \in E, u \diamond e = e \diamond u = u$
3) Symétri~~que~~ $\forall u \in G \quad \exists u' \in G, \; u \diamond u' = u' \diamond u = e$

4) Commutative $\forall u, v \in E, \; u \diamond v = v \diamond u$

Exercice 1 :

0) $G =]-1, 1[$

$\Delta : G \times G \longmapsto G$? } on teste en remplaçant x et y par $]-1,1[$

$(x, y) \longmapsto \frac{x+y}{1+x \cdot y}$

On suppose
$-1 < x < 1$
$-1 < y < 1$

On veut montrer

$$-1 < \frac{x+y}{1+x \cdot y} < 1 \iff \left|\frac{x+y}{1+x \cdot y}\right| < 1 \iff \left(\frac{x+y}{1+x y}\right)^2 < 1$$

$$\iff \frac{(x+y)^2}{(1+x y)^2} < 1$$

$$\iff (x+y)^2 < (1+x y)^2$$

$$\iff 0 < (1+xy)^2 - (x+y)^2$$

$$\iff 0 < 1 + 2xy + x^2y^2 - (x^2 + 2xy + y^2)$$

$$\iff 0 < 1 + 2xy + x^2y^2 - x^2 - 2xy - y^2 = x^2y^2 - x^2 + 1 - y^2$$
$$= -x^2(1-y^2) + 1 - y^2 = \underset{>0}{(1-x^2)}\underset{>0}{(1-y^2)}$$

1) $x \Delta y = \frac{x+y}{1+x y}$

$\forall (x, y, z) \in G^3$
$\forall x, y, z \in G$

$$(x \Delta y) \Delta z = \left(\frac{x+y}{1+(x \cdot y)}\right) \Delta z = \frac{\frac{x+y}{1+xy} + z}{1 + \left(\frac{x+y}{1+xy}\right) z} \times \frac{1+xy}{1+xy} = \frac{\left(\frac{x+y}{1+xy} + z\right)(1+xy)}{\left[1+\left(\frac{x+y}{1+xy}\right) z\right]\left[1+xy\right]}$$

$$= \frac{x+y+z+xyz}{1+xy+yz+zx}$$

$$x \Delta (y \Delta z) = x \Delta \frac{y+z}{1+yz} = \frac{x + \frac{y+z}{1+y \cdot z}}{1 + x \frac{y+z}{1+y \cdot z}} \times \frac{1+yz}{1+yz} = \frac{y+z+x+xyz}{1+yz+xy+xz}$$