Chapter 7
Response of First-Order *RL* and *RC* Circuits

Learning Goals for Chapter 7

- Be able to calculate the complete response for first-order *RL* and *RC* circuits with constant inputs

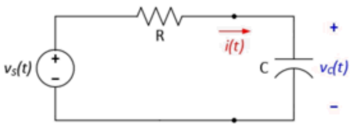
Complete Response of RC and RL Circuits

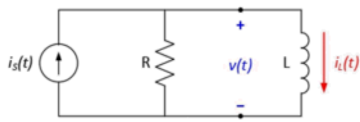
- Voltages and currents in circuits that contain capacitors and inductors are solutions to differential equations.

- The order of the differential equation is usually equal to the total number of capacitors and inductors in the circuit.

- Circuits that contain only one capacitor or inductor can be represented by a first-order differential equation and are called first-order circuits.

- All first-order circuits are equivalent to one of the following:





$$v_C(t) + RC\frac{dv_C(t)}{dt} = v_S(t)$$

$$i_L(t) + \frac{L}{R}\frac{di_L(t)}{dt} = i_S(t)$$

$$\frac{dv_C(t)}{dt} + \frac{v_C(t)}{RC} = \frac{v_S(t)}{RC}$$

$$\frac{di_L(t)}{dt} + \frac{R}{L}i_L(t) = \frac{R}{L}i_S(t)$$

- These first-order differential equations have the same form:

$$\frac{dx(t)}{dt} + \frac{x(t)}{\tau} = \frac{y(t)}{\tau}$$

<u>RC Circuit</u>

$y(t) = v_S(t)$

$x(t) = v_C(t)$

$\tau = RC$

<u>RL Circuit</u>

$y(t) = i_S(t)$

$x(t) = i_L(t)$

$\tau = \frac{L}{R}$

- The constant τ is called the time constant.

- The solution to this first-order differential equation can be expressed in several equivalent forms:

complete response = *transient response* + *steady-state response*

complete response = *natural response* + *forced response*

complete response = *homogeneous response* + *particular solution*

- We will only consider first-order circuits with constant inputs.

- Solution approach:
  - Determine the initial condition of the energy storage element.
  - Determine the steady-state or forced response.
  - Add the natural response to the forced response to obtain the complete response. Use the initial condition to resolve the constant in the natural response.

Constant Input

$$y(t) = M = x(\infty)$$

Then:

$$\frac{dx(t)}{dt} + \frac{x(t)}{\tau} = \frac{M}{\tau}$$

$$\frac{dx(t)}{dt} = \frac{M - x(t)}{\tau}$$

$$\frac{dx(t)}{x(t) - M} = -\frac{dt}{\tau}$$

$$\int \frac{dx}{x - M} = -\frac{1}{\tau}\int dt$$

$$\ln(x - M) = -\frac{t}{\tau} + D$$

$$e^{\ln(x-M)} = e^{\frac{-t}{\tau}+D}$$

$$\mathrm{x} - \mathrm{M} = Ke^{\frac{-t}{\tau}} \qquad \text{where} \qquad K = e^{D}$$

So the complete response is:

$$x(t) = M + Ke^{\frac{-t}{\tau}}$$

At time $t = 0$:

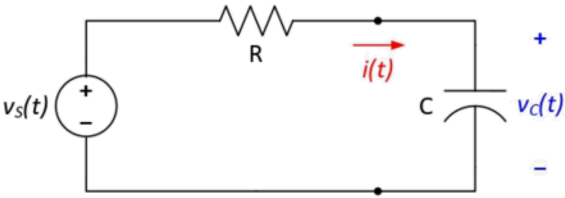
$$x(0) = M + K$$

$$K = x(0) - M = x(0) - x(\infty)$$

Finally:

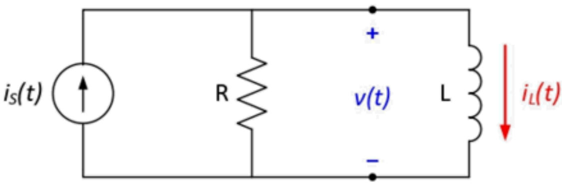
$$x(t) = x(\infty) + [x(0) - x(\infty)]e^{\frac{-t}{\tau}}$$

- For the original RC circuit with $v_S(t) = M = x(\infty) = V_{OC}$:



$$v_C(t) = V_{OC} + [v_C(0) - V_{OC}]e^{\frac{-t}{RC}}$$

- For the original RL circuit with $i_S(t) = M = x(\infty) = I_{SC}$:



$$i_L(t) = I_{SC} + [i_L(0) - I_{SC}]e^{\frac{-R}{L}t}$$

- These results can be shifted to time $t = t_0$:

$$v_C(t) = V_{OC} + [v_C(t_0) - V_{OC}]e^{\frac{-(t-t_0)}{RC}} \qquad \text{for } t > t_0$$

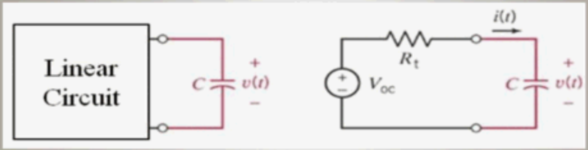
$$i_L(t) = I_{SC} + [i_L(t_0) - I_{SC}]e^{\frac{-R}{L}(t-t_0)} \qquad \text{for } t > t_0$$

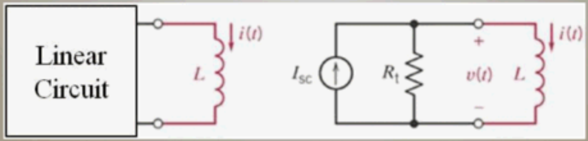
First-order Circuits

- Circuits that contain only one energy storage element (a capacitor or an inductor) and resistors.
- Circuits may be energized by voltage sources and/or current sources.
- We will consider only circuits with constant (DC) voltage or current sources.

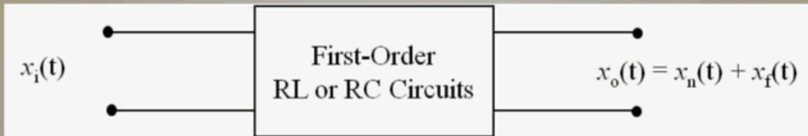
First-order Circuits

- Any first-order circuit can be reduced to either a Thévenin equivalent or a Norton equivalent circuit of the following forms:





First-order Circuits



$$x_0(t) = x_n(t) + x_f(t)$$

- x_0(t): complete response
- x_n(t): natural, transient or homogeneous response
- x_f(t): forced, steady-state or particular response

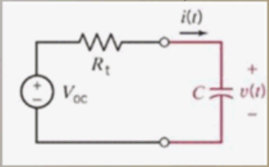
Complete Response to a Constant Input

- The complete response of the circuit is

$$x(t) = x(\infty) + \left[x(0) - x(\infty)\right]e^{-t/\tau}$$

- For a *RC* circuit ***(find Thévenin Equivalent Circuit)***

$$v_C(t) = v_{oc} + \left[v_C(0) - v_{oc}\right]e^{-t/\tau} \quad where \quad \tau = R_{th}C$$



- For a *RL* circuit ***(find Norton Equivalent Circuit)***,

$$i_L(t) = i_{sc} + \left[i_L(0) - i_{sc}\right]e^{-t/\tau} \quad where \quad \tau = \frac{L}{R_{th}}$$