# 100A HW6 Solutions

**Problem 1** Given $x \in \{a, b, c\}$ with prior $p(x)$ and $y \in \{1, 2, 3, 4\}$ with conditional $p(y|x)$:

1. **Joint Distribution $p(x, y)$:** Use $p(x, y) = p(x) \cdot p(y|x)$.

$$p(x = a, y = 1) = 0.4 \cdot 0.20 = 0.08$$

$$p(x = a, y = 2) = 0.4 \cdot 0.30 = 0.12$$

$$p(x = a, y = 3) = 0.4 \cdot 0.40 = 0.16$$

$$p(x = a, y = 4) = 0.4 \cdot 0.10 = 0.04$$

$$p(x = b, y = 1) = 0.3 \cdot 0.50 = 0.15$$

$$p(x = b, y = 2) = 0.3 \cdot 0.20 = 0.06$$

$$p(x = b, y = 3) = 0.3 \cdot 0.20 = 0.06$$

$$p(x = b, y = 4) = 0.3 \cdot 0.10 = 0.03$$

$$p(x = c, y = 1) = 0.3 \cdot 0.10 = 0.03$$

$$p(x = c, y = 2) = 0.3 \cdot 0.10 = 0.03$$

$$p(x = c, y = 3) = 0.3 \cdot 0.30 = 0.09$$

$$p(x = c, y = 4) = 0.3 \cdot 0.50 = 0.15$$

| | $y = 1$ | $y = 2$ | $y = 3$ | $y = 4$ |
|---|---|---|---|---|
| $x = a$ | 0.08 | 0.12 | 0.16 | 0.04 |
| $x = b$ | 0.15 | 0.06 | 0.06 | 0.03 |
| $x = c$ | 0.03 | 0.03 | 0.09 | 0.15 |

Table 1: Joint distribution $p(x, y)$.

2. **Marginal Distribution $p(y)$:** Sum over $x$ for each $y$:

$$p(y = 1) = 0.08 + 0.15 + 0.03 = 0.26$$

$$p(y = 2) = 0.12 + 0.06 + 0.03 = 0.21$$

$$p(y = 3) = 0.16 + 0.06 + 0.09 = 0.31$$

$$p(y = 4) = 0.04 + 0.03 + 0.15 = 0.22$$

Check: $0.26 + 0.21 + 0.31 + 0.22 = 1$.

3. **Backward Conditional Distribution** $p(x|y)$**:** Use $p(x|y) = \frac{p(x,y)}{p(y)}$.

$$p(x = a|y = 1) = \frac{0.08}{0.26} \approx 0.3077$$

$$p(x = b|y = 1) = \frac{0.15}{0.26} \approx 0.5769$$

$$p(x = c|y = 1) = \frac{0.03}{0.26} \approx 0.1154$$

$$p(x = a|y = 2) = \frac{0.12}{0.21} \approx 0.5714$$

$$p(x = b|y = 2) = \frac{0.06}{0.21} \approx 0.2857$$

$$p(x = c|y = 2) = \frac{0.03}{0.21} \approx 0.1429$$

$$p(x = a|y = 3) = \frac{0.16}{0.31} \approx 0.5161$$

$$p(x = b|y = 3) = \frac{0.06}{0.31} \approx 0.1935$$

$$p(x = c|y = 3) = \frac{0.09}{0.31} \approx 0.2903$$

$$p(x = a|y = 4) = \frac{0.04}{0.22} \approx 0.1818$$

$$p(x = b|y = 4) = \frac{0.03}{0.22} \approx 0.1364$$

$$p(x = c|y = 4) = \frac{0.15}{0.22} \approx 0.6818$$

| | $y = 1$ | $y = 2$ | $y = 3$ | $y = 4$ |
|---|---|---|---|---|
| $x = a$ | 0.3077 | 0.5714 | 0.5161 | 0.1818 |
| $x = b$ | 0.5769 | 0.2857 | 0.1935 | 0.1364 |
| $x = c$ | 0.1154 | 0.1429 | 0.2903 | 0.6818 |

Table 2: Backward conditional distribution $p(x|y)$ (rounded to 4 decimals).

## Problem 2

1. **Density of** $X \sim \mathcal{N}(\mu, \sigma^2)$**:**

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

2. **Conditional Density** $f(y|x)$**:** Given $X \sim \mathcal{N}(0, 1)$ and $[Y|X = x] \sim \mathcal{N}(\rho x, 1 - \rho^2)$:

$$f(y|x) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(y-\rho x)^2}{2(1-\rho^2)}\right)$$

3. **Conditional Expectation and Variance:**

   (a) $\mathrm{E}[Y|X = x] = \rho x$ (mean of $\mathcal{N}(\rho x, 1 - \rho^2)$).

   (b) $\mathrm{Var}[Y|X = x] = 1 - \rho^2$ (variance of $\mathcal{N}(\rho x, 1 - \rho^2)$).

4. **Joint Density $f(x, y)$:** Use $f(x, y) = f(x)f(y|x)$, where $f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$:

$$f(x, y) = \left(\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\right)\left(\frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(y-\rho x)^2}{2(1-\rho^2)}\right)\right)$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2}{2} - \frac{(y-\rho x)^2}{2(1-\rho^2)}\right)$$

Expand the exponent:

$$-\frac{x^2}{2} - \frac{y^2 - 2\rho xy + \rho^2 x^2}{2(1-\rho^2)} = -\frac{x^2(1-\rho^2) + y^2 - 2\rho xy + \rho^2 x^2}{2(1-\rho^2)}$$

$$= -\frac{x^2 + y^2 - 2\rho xy}{2(1-\rho^2)}$$

Thus:

$$f(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right)$$

**Problem 3**

(1) Given $X_i \sim \mathrm{Uniform}[0, 1]$ independently, $i = 1, \ldots, n$:

$$\mu = \mathrm{E}(X_i) = \int_0^1 x \cdot 1\, dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

$$\mathrm{E}(X_i^2) = \int_0^1 x^2 \cdot 1\, dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\sigma^2 = \mathrm{Var}(X_i) = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

For $S = \sum_{i=1}^n X_i$:

$$\mathrm{E}(S) = \sum_{i=1}^n \mathrm{E}(X_i) = n \cdot \frac{1}{2} = \frac{n}{2}$$

$$\mathrm{Var}(S) = \sum_{i=1}^n \mathrm{Var}(X_i) = n \cdot \frac{1}{12} = \frac{n}{12} \quad \text{(independence)}$$

For $\bar{X} = S/n$:

$$\mathrm{E}(\bar{X}) = \mathrm{E}\left(\frac{S}{n}\right) = \frac{\mathrm{E}(S)}{n} = \frac{n/2}{n} = \frac{1}{2}$$

$$\mathrm{Var}(\bar{X}) = \mathrm{Var}\left(\frac{S}{n}\right) = \frac{\mathrm{Var}(S)}{n^2} = \frac{n/12}{n^2} = \frac{1}{12n}$$

By the Central Limit Theorem (CLT), for large $n$:

$$S \approx \mathcal{N}\left(\frac{n}{2}, \frac{n}{12}\right)$$

$$\bar{X} \approx \mathcal{N}\left(\frac{1}{2}, \frac{1}{12n}\right)$$

(2) Given $Z_i \sim \text{Bernoulli}(p)$ independently, $i = 1, \ldots, n$:

$$\mu = \mathrm{E}(Z_i) = p \cdot 1 + (1-p) \cdot 0 = p$$

$$\mathrm{E}(Z_i^2) = p \cdot 1^2 + (1-p) \cdot 0 = p$$

$$\sigma^2 = \mathrm{Var}(Z_i) = p - p^2 = p(1-p)$$

For $X = \sum_{i=1}^{n} Z_i$:

$$\mathrm{E}(X) = \sum_{i=1}^{n} \mathrm{E}(Z_i) = np$$

$$\mathrm{Var}(X) = \sum_{i=1}^{n} \mathrm{Var}(Z_i) = np(1-p) \quad \text{(independence)}$$

For $X/n$:

$$\mathrm{E}(X/n) = \frac{\mathrm{E}(X)}{n} = \frac{np}{n} = p$$

$$\mathrm{Var}(X/n) = \frac{\mathrm{Var}(X)}{n^2} = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}$$

Exact distribution of $X$: $X \sim \text{Binomial}(n, p)$ (sum of independent Bernoullis). Approximate distribution of $X/n$: By CLT, for large $n$:

$$X/n \approx \mathcal{N}\left(p, \frac{p(1-p)}{n}\right)$$