STATS 100A HW3 Solutions

Problem 1

For random variables X and Y, the rules are:

1. $p(y) = \sum_x p(x, y)$
2. $p(x, y) = p(x)p(y|x)$
3. $p(x|y) = \frac{p(x,y)}{p(y)}$

Explanation

(1) Population Proportion:
Assume a population of N people:

- $N(x, y)$: number of people with $X = x$ and $Y = y$, so $p(x, y) = \frac{N(x,y)}{N}$,
- $N(x)$: number of people with $X = x$, so $p(x) = \frac{N(x)}{N}$,
- $N(y)$: number of people with $Y = y$, so $p(y) = \frac{N(y)}{N}$.
- **Rule 1:** $N(y) = \sum_x N(x, y)$, thus $p(y) = \frac{N(y)}{N} = \sum_x \frac{N(x,y)}{N} = \sum_x p(x, y)$.
- **Rule 2:** $N(x, y) = N(x) \cdot \frac{N(x,y)}{N(x)}$, so $p(x, y) = \frac{N(x,y)}{N} = \frac{N(x)}{N} \cdot \frac{N(x,y)}{N(x)} = p(x)p(y|x)$, where $p(y|x) = \frac{N(x,y)}{N(x)}$.
- **Rule 3:** $\frac{N(x,y)}{N(y)} = \frac{\frac{N(x,y)}{N}}{\frac{N(y)}{N}}$, so $p(x|y) = \frac{N(x,y)}{N(y)} = \frac{p(x,y)}{p(y)}$.

(2) Long Run Frequency:
Assume n repetitions:

- $n(x, y)$: number of times $X = x$ and $Y = y$ occur, so $p(x, y) \approx \frac{n(x,y)}{n}$ (fluctuates, approaches true value as $n \to \infty$),
- $n(x)$: number of times $X = x$ occurs, so $p(x) \approx \frac{n(x)}{n}$,
- $n(y)$: number of times $Y = y$ occurs, so $p(y) \approx \frac{n(y)}{n}$.
- **Rule 1:** $n(y) = \sum_x n(x, y)$, thus $p(y) \approx \frac{n(y)}{n} = \sum_x \frac{n(x,y)}{n} = \sum_x p(x, y)$.
- **Rule 2:** $n(x, y) \approx n(x) \cdot \frac{n(x,y)}{n(x)}$, so $p(x, y) \approx \frac{n(x,y)}{n} = \frac{n(x)}{n} \cdot \frac{n(x,y)}{n(x)} = p(x)p(y|x)$, where $p(y|x) \approx \frac{n(x,y)}{n(x)}$.
- **Rule 3:** $\frac{n(x,y)}{n(y)} = \frac{\frac{n(x,y)}{n}}{\frac{n(y)}{n}}$, so $p(x|y) \approx \frac{n(x,y)}{n(y)} = \frac{p(x,y)}{p(y)}$.

Example

Let X = disease (0 = no, 1 = yes), Y = symptom (0 = no, 1 = yes), with:

- $p(X=0) = 0.9$, $p(X=1) = 0.1$,
- $p(Y=1|X=0) = 0.05$, $p(Y=0|X=0) = 0.95$,
- $p(Y=1|X=1) = 0.8$, $p(Y=0|X=1) = 0.2$.

Joint Probabilities (Rule 2):

$$p(X=0,Y=0) = 0.9 \cdot 0.95 = 0.855, \quad p(X=0,Y=1) = 0.9 \cdot 0.05 = 0.045$$

$$p(X=1,Y=0) = 0.1 \cdot 0.2 = 0.02, \quad p(X=1,Y=1) = 0.1 \cdot 0.8 = 0.08$$

$X\backslash Y$	0	1
0	0.855	0.045
1	0.02	0.08

Table 1: Joint probabilities $p(x,y)$

Marginal $p(y)$ (Rule 1):

$$p(Y=0) = 0.855 + 0.02 = 0.875, \quad p(Y=1) = 0.045 + 0.08 = 0.125$$

Conditional $p(x|y)$ (Rule 3):

$$p(X=0|Y=0) = \frac{0.855}{0.875} \approx 0.977, \quad p(X=1|Y=0) = \frac{0.02}{0.875} \approx 0.023$$

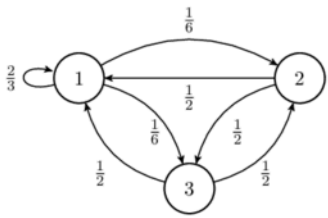
$$p(X=0|Y=1) = \frac{0.045}{0.125} = 0.36, \quad p(X=1|Y=1) = \frac{0.08}{0.125} = 0.64$$

$X\backslash Y$	0	1
0	0.977	0.36
1	0.023	0.64

Table 2: Conditional probabilities $p(x|y)$

Problem 2

A:



(1)

$$K = \begin{pmatrix} 2/3 & 1/6 & 1/6 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

(2) Start from webpage 1, $p^{(0)} = \begin{pmatrix} 1, 0, 0 \end{pmatrix}$.

Since $p^{(t)} = p^{(t-1)}K$,

$$p^{(1)} = p^{(0)}K = \begin{pmatrix} 1, 0, 0 \end{pmatrix} \begin{pmatrix} 2/3 & 1/6 & 1/6 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \left(\tfrac{2}{3}, \tfrac{1}{6}, \tfrac{1}{6} \right)$$

$$p^{(2)} = p^{(1)}K = \left(\tfrac{2}{3}, \tfrac{1}{6}, \tfrac{1}{6} \right) \begin{pmatrix} 2/3 & 1/6 & 1/6 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \left(\tfrac{11}{18}, \tfrac{7}{36}, \tfrac{7}{36} \right)$$

$$p^{(3)} = p^{(2)}K = \left(\tfrac{11}{18}, \tfrac{7}{36}, \tfrac{7}{36} \right) \begin{pmatrix} 2/3 & 1/6 & 1/6 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \left(\tfrac{65}{108}, \tfrac{43}{216}, \tfrac{43}{216} \right)$$

(3)Solving $\pi K = \pi$ with constraint $\sum_{i=1}^{3} \pi_i = 1$ is equivalent to solving a system of linear equations:

$$\begin{cases} \frac{2}{3}\pi_1 + \frac{1}{2}\pi_2 + \frac{1}{2}\pi_3 = \pi_1 \\ \frac{1}{6}\pi_1 + \frac{1}{2}\pi_3 = \pi_2 \\ \frac{1}{6}\pi_1 + \frac{1}{2}\pi_2 = \pi_3 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

Then $\pi = \left(\frac{3}{5}, \frac{1}{5}, \frac{1}{5} \right)$, which is close to $p^{(3)}$.

(4) On average, the distributions for one million people at time $t (t = 1, 2, 3)$ resemble the $p^{(t)}$ in (2). The stationary distribution is π in (3), showing that webpage 1 is most popular.

Problem 3

(1) Let A be the event of alarm, and F be the event of fire.

$$\begin{aligned} P(F|A) &= \frac{P(F \cap A)}{P(A)} \\ &= \frac{P(F)P(A|F)}{P(F)P(A|F) + P(F^c)P(A|F^c)} \\ &= \frac{\alpha\beta}{\alpha\beta + (1-\alpha)\gamma}. \end{aligned}$$

(2) Suppose $\alpha = 1/1000$, and $\beta = 99/100$, and $\gamma = 2/100$. Then

$$\begin{aligned} P(F|A) &= \frac{1/1000 \times 99/100}{1/1000 \times 99/100 + (1 - 1/1000) \times 2/100} \\ &= \frac{99}{99 + 999 \times 2} = .047. \end{aligned}$$

Imagine we repeat the experiments 100,000 times. Then on average, 100 times there is fire. Out of these 100 times, 99 times there is alarm. However, there are 99900 times there is no fire. Out of these 99900 times, 999×2 times there is alarm. In total, there will be $(99 + 999 \times 2)$ times there is alarm, and among these times, only 99 times there is fire. So $P(F|A) = 99/(99 + 999 \times 2)$.

Problem 4: Bayes Net Summary

- A Bayes net (Bayesian network) is a graphical model representing probabilistic relationships among variables using a directed acyclic graph (DAG).
- Nodes represent random variables, and edges indicate conditional dependencies.
- Each node has a conditional probability distribution given its parents.
- It efficiently encodes joint distributions, enabling inference (e.g., updating beliefs) via Bayes' rule.
- Applications include diagnostics and decision-making under uncertainty.

STATS 100A HW4 Solutions

Problem 1

Suppose we roll a die with $p(1) = 0.1$, $p(2) = 0.1$, $p(3) = 0.1$, $p(4) = 0.2$, $p(5) = 0.2$, $p(6) = 0.3$.

(1) Probabilities

- $P(X > 4) = P(X = 5) + P(X = 6) = 0.2 + 0.3 = 0.5$
- $P(X = 6|X > 4) = \frac{P(X=6 \text{ and } X>4)}{P(X>4)} = \frac{P(X=6)}{P(X>4)} = \frac{0.3}{0.5} = 0.6$

(2) Expectation, Variance, Standard Deviation

- $\mathrm{E}(X) = \sum xp(x) = 1\cdot 0.1+2\cdot 0.1+3\cdot 0.1+4\cdot 0.2+5\cdot 0.2+6\cdot 0.3 = 0.1+0.2+0.3+0.8+1.0+1.8 = 4.2$
- $\mathrm{E}(X^2) = \sum x^2p(x) = 1^2 \cdot 0.1 + 2^2 \cdot 0.1 + 3^2 \cdot 0.1 + 4^2 \cdot 0.2 + 5^2 \cdot 0.2 + 6^2 \cdot 0.3 = 0.1 + 0.4 + 0.9 + 3.2 + 5.0 + 10.8 = 20.4$
- $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = 20.4 - 4.2^2 = 20.4 - 17.64 = 2.76$
- $SD(X) = \sqrt{\mathrm{Var}(X)} = \sqrt{2.76} \approx 1.661$

(3) Reward Function $h(X)$

Given $h(1) = -20$, $h(2) = -10$, $h(3) = 0$, $h(4) = 10$, $h(5) = 20$, $h(6) = 100$:

- $\mathrm{E}(h(X)) = \sum h(x)p(x) = -20 \cdot 0.1 + (-10) \cdot 0.1 + 0 \cdot 0.1 + 10 \cdot 0.2 + 20 \cdot 0.2 + 100 \cdot 0.3 = -2 - 1 + 0 + 2 + 4 + 30 = 33$ dollars
- $\mathrm{E}(h(X)^2) = (-20)^2 \cdot 0.1 + (-10)^2 \cdot 0.1 + 0^2 \cdot 0.1 + 10^2 \cdot 0.2 + 20^2 \cdot 0.2 + 100^2 \cdot 0.3 = 40 + 10 + 0 + 20 + 80 + 3000 = 3150$ dollars^2
- $\mathrm{Var}(h(X)) = \mathrm{E}(h(X)^2) - [\mathrm{E}(h(X))]^2 = 3150 - 33^2 = 3150 - 1089 = 2061$ dollars^2
- $SD(h(X)) = \sqrt{2061} \approx 45.399$ dollars
- Units: $\mathrm{E}(h(X))$ in dollars, $\mathrm{Var}(h(X))$ in dollars squared

(4) Linear Function $h(x) = 2x + 1$

- $\mathrm{E}(h(X)) = \sum h(x)p(x) = (2\cdot 1+1)\cdot 0.1+(2\cdot 2+1)\cdot 0.1+(2\cdot 3+1)\cdot 0.1+(2\cdot 4+1)\cdot 0.2+(2\cdot 5+1)\cdot 0.2+(2\cdot 6+1)\cdot 0.3 = 3\cdot 0.1+5\cdot 0.1+7\cdot 0.1+9\cdot 0.2+11\cdot 0.2+13\cdot 0.3 = 0.3+0.5+0.7+1.8+2.2+3.9 = 9.4$
- Verify: $2\mathrm{E}(X) + 1 = 2 \cdot 4.2 + 1 = 8.4 + 1 = 9.4$, which matches.

(5) Quadratic Function $h(x) = x^2$

- $\mathrm{E}(h(X)) = \mathrm{E}(X^2) = \sum x^2 p(x) = 20.4$ (from (2))
- Verify: $\mathrm{E}(X^2) = 20.4 \geq [\mathrm{E}(X)]^2 = 4.2^2 = 17.64$, which holds (Jensen's inequality).

Problem 2

$\mathrm{E}(Z) = 1 \times p + 0 \times (1-p) = p$. Because $Z^2 = Z$, $\mathrm{E}(Z^2) = \mathrm{E}(Z) = p$. $\mathrm{Var}(Z) = \mathrm{E}(Z^2) - \mathrm{E}(Z)^2 = p - p^2 = p(1-p)$. If $p = 1/2$, $\mathrm{E}(Z) = 1/2$ and $\mathrm{Var}(Z) = 1/4$.

If we replace 0 by -1, we have $\mathrm{E}(Z) = 1 \times p + (-1) * (1-p) = p - (1-p) = 2p - 1$. $\mathrm{E}(Z^2) = 1^2 \times p + (-1)^2 \times (1-p) = 1$. $\mathrm{Var}(Z) = \mathrm{E}(Z^2) - \mathrm{E}(Z)^2 = 1 - (2p-1)^2 = 4p(1-p)$. If $p = 1/2$, $\mathrm{E}(Z) = 0$, $\mathrm{Var}(Z) = 1$.

Problem 3

Flip a fair coin 100 times, $X \sim \mathrm{Binomial}(100, 1/2)$:

$$\begin{aligned}
&\mathrm{E}(X) = np = 100 \times (1/2) = 50 \\
&\mathrm{Var}(X) = np(1-p) = 100 \times (1/2) \times (1 - 1/2) = 25 \\
&SD(X) = \sqrt{\mathrm{Var}(X)} = 5 \\
&\mathrm{E}(X/100) = \mathrm{E}(X)/100 = 1/2 \\
&\mathrm{Var}(X/100) = \mathrm{Var}(X)/100^2 = 2.5 \times 10^{-3} \\
&SD(X/100) = SD(X)/100 = .05 \\
&P(X \in [40, 60]) = \sum_{k=40}^{60} P(X = k) = \sum_{k=40}^{60} \binom{100}{k} / 2^{100}
\end{aligned}$$

Problem 4

Sample 100 voters with replacement, 20% support A, $X \sim \mathrm{Binomial}(100, 0.2)$:

$$\begin{aligned}
&X \sim \mathrm{Binomial}(n = 100, p = .2) \\
&\mathrm{E}(X) = np = 100 \times .2 = 20 \\
&\mathrm{Var}(X) = np(1-p) = 100 \times .2 \times (1 - .2) = 16 \\
&SD(X) = \sqrt{\mathrm{Var}(X)} = 4 \\
&\mathrm{E}(X/100) = \mathrm{E}(X)/100 = .2 \\
&\mathrm{Var}(X/100) = \mathrm{Var}(X)/100^2 = 1.6 \times 10^{-3} \\
&SD(X/100) = SD(X)/100 = .04
\end{aligned}$$

Problem 5

Throw 10,000 points into $[0,1]^2$, $A = \{x^2 + y^2 \leq 1\}$, m is the number in A, $\hat{\pi} = 4m/10,000$:

$$m \sim \text{Binomial}(n = 10^4, p = \pi/4)$$
$$\mathrm{E}(\hat{\pi}) = \mathrm{E}(4m/10^4) = (4/10^4)\mathrm{E}(m) = (4/10^4) \times 10^4 \times \pi/4 = \pi$$
$$\mathrm{Var}(\hat{\pi}) = \mathrm{Var}(4m/10^4) = (4/10^4)^2\mathrm{Var}(m) = (4/10^4)^2 \times 10^4 \times \pi/4 \times (1 - \pi/4) = \pi(4-\pi)/10^4$$
$$SD(\hat{\pi}) = \sqrt{\pi(4-\pi)}/10^2$$

100A HW5 Solutions

Problem 1 For a continuous random variable X, let $f(x) = ax^2$ for $x \in [0,1]$, and $f(x) = 0$ otherwise, where a is a constant.

(1) Since $f(x)$ is a density function, it must integrate to 1 over its support:

$$\int_0^1 ax^2\,dx = a\int_0^1 x^2\,dx = a\left[\frac{x^3}{3}\right]_0^1 = a \cdot \frac{1}{3} = 1$$

Thus, $a = 3$. So, $f(x) = 3x^2$ for $x \in [0,1]$.

(2) Compute $P(X \geq 1/2)$:

$$P(X \geq 1/2) = \int_{1/2}^1 3x^2\,dx = \left[x^3\right]_{1/2}^1 = 1 - \left(\frac{1}{2}\right)^3 = 1 - \frac{1}{8} = \frac{7}{8}$$

The CDF $F(x) = P(X \leq x)$ for $x \in [0,1]$ is:

$$F(x) = \int_0^x 3t^2\,dt = \left[t^3\right]_0^x = x^3$$

So, $F(x) = x^3$ for $x \in [0,1]$, $F(x) = 0$ for $x < 0$, and $F(x) = 1$ for $x > 1$.

(3) Calculate the moments:

$$\mathrm{E}(X) = \int_0^1 x \cdot 3x^2\,dx = 3\int_0^1 x^3\,dx = 3\left[\frac{x^4}{4}\right]_0^1 = 3 \cdot \frac{1}{4} = \frac{3}{4}$$

$$\mathrm{E}(X^2) = \int_0^1 x^2 \cdot 3x^2\,dx = 3\int_0^1 x^4\,dx = 3\left[\frac{x^5}{5}\right]_0^1 = 3 \cdot \frac{1}{5} = \frac{3}{5}$$

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = \frac{3}{5} - \frac{9}{16} = \frac{48}{80} - \frac{45}{80} = \frac{3}{80}$$

(4) The scatterplot would show points along $[0,1]$ with density increasing toward 1, and the histogram would resemble the curve $3x^2$, denser near 1.

Problem 2 Suppose $T \sim \text{Exponential}(\lambda)$, with density $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$ and $f(t) = 0$ for $t < 0$.

(1) The CDF is:

$$F(t) = P(T \leq t) = \int_0^t \lambda e^{-\lambda s}\,ds = \left[-e^{-\lambda s}\right]_0^t = -e^{-\lambda t} - (-e^0) = 1 - e^{-\lambda t} \quad (t \geq 0), \quad F(t) = 0\,(t < 0)$$

Set $F(t_{\text{half}}) = 1/2$:

$$1 - e^{-\lambda t_{\text{half}}} = \frac{1}{2} \implies e^{-\lambda t_{\text{half}}} = \frac{1}{2} \implies -\lambda t_{\text{half}} = \ln\frac{1}{2} \implies t_{\text{half}} = \frac{\ln 2}{\lambda}$$

(2) Using integration by parts ($u = t, dv = \lambda e^{-\lambda t}dt$):

$$\mathrm{E}(T) = \int_0^\infty t \cdot \lambda e^{-\lambda t}\,dt = \left[-te^{-\lambda t}\right]_0^\infty + \int_0^\infty e^{-\lambda t}\,dt = 0 + \left[-\frac{1}{\lambda}e^{-\lambda t}\right]_0^\infty = \frac{1}{\lambda}$$

$$\mathrm{E}(T^2) = \int_0^\infty t^2 \cdot \lambda e^{-\lambda t}\, dt = \left[-t^2 e^{-\lambda t}\right]_0^\infty + 2\int_0^\infty t e^{-\lambda t}\, dt = 0 + \frac{2}{\lambda}\cdot\frac{1}{\lambda} = \frac{2}{\lambda^2}$$

$$\mathrm{Var}(T) = \mathrm{E}(T^2) - [\mathrm{E}(T)]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

Problem 3 Suppose $U \sim \mathrm{Uniform}[0,1]$.

(1) For $X = U^2$, find $F(x) = P(X \le x) = P(U^2 \le x) = P(U \le \sqrt{x})$ (since $U \ge 0$). The CDF of U is $P(U \le u) = u$ for $u \in [0,1]$, so:

$$F(x) = P(U \le \sqrt{x}) = \sqrt{x} \quad (0 \le x \le 1), \quad F(x) = 0\ (x < 0), \quad F(x) = 1\ (x > 1)$$

The PDF is:

$$f(x) = F'(x) = \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}} \quad (0 < x \le 1), \quad f(x) = 0 \text{ otherwise}$$

(2) For $T = -\log U$, find $F(t) = P(T \le t) = P(-\log U \le t) = P(\log U \ge -t) = P(U \ge e^{-t})$. Since $P(U \ge u) = 1 - u$ for $u \in [0,1]$:

$$F(t) = P(U \ge e^{-t}) = 1 - e^{-t} \quad (t \ge 0), \quad F(t) = 0\ (t < 0)$$

The PDF is:

$$f(t) = F'(t) = \frac{d}{dt}(1 - e^{-t}) = e^{-t} \quad (t \ge 0), \quad f(t) = 0\ (t < 0)$$

Problem 4 Suppose $Z \sim N(0,1)$, with density $f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$.

(1) Calculate $\mathrm{E}(Z)$:

$$\mathrm{E}(Z) = \int_{-\infty}^\infty z \cdot \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

The integrand $ze^{-z^2/2}$ is an odd function, and the limits are symmetric, so:

$$\int_{-\infty}^\infty z \cdot \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz = 0$$

For $\mathrm{E}(Z^2)$, use integration by parts ($u = z, dv = ze^{-z^2/2} \cdot \frac{1}{\sqrt{2\pi}} dz$):

$$dv = ze^{-z^2/2} \cdot \frac{1}{\sqrt{2\pi}} dz, \quad v = \int ze^{-z^2/2} \cdot \frac{1}{\sqrt{2\pi}} dz = -\frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

$$\mathrm{E}(Z^2) = \int_{-\infty}^\infty z^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz = \left[z\left(-\frac{1}{\sqrt{2\pi}} e^{-z^2/2}\right)\right]_{-\infty}^\infty + \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

The boundary term: $\lim_{z\to\infty} -ze^{-z^2/2}/\sqrt{2\pi} = 0$ (exponential decay dominates), and similarly at $-\infty$, so it's 0. The remaining integral is the density's total probability:

$$\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz = 1$$

Thus, $\mathrm{E}(Z^2) = 1$, and:

$$\mathrm{Var}(Z) = \mathrm{E}(Z^2) - [\mathrm{E}(Z)]^2 = 1 - 0 = 1$$

(2) For $X = \mu + \sigma Z$:

$$\mathrm{E}(X) = \mathrm{E}(\mu + \sigma Z) = \mu + \sigma \mathrm{E}(Z) = \mu$$

$$\mathrm{Var}(X) = \mathrm{Var}(\mu + \sigma Z) = \sigma^2 \mathrm{Var}(Z) = \sigma^2$$

(3) Since $X = \mu + \sigma Z$, $Z = \frac{X-\mu}{\sigma}$. Consider probabilities over small intervals:

$$P(X \in (x, x + \Delta x)) = P(\mu + \sigma Z \in (x, x + \Delta x)) = P\left(Z \in \left(\frac{x-\mu}{\sigma}, \frac{x + \Delta x - \mu}{\sigma}\right)\right)$$

Let $z = \frac{x-\mu}{\sigma}$, $\Delta z = \frac{\Delta x}{\sigma}$, so:

$$P(Z \in (z, z + \Delta z)) = f(z)\Delta z, \quad P(X \in (x, x + \Delta x)) = g(x)\Delta x$$

Equate: $g(x)\Delta x = f(z)\Delta z = f\left(\frac{x-\mu}{\sigma}\right) \cdot \frac{\Delta x}{\sigma}$, thus:

$$g(x) = f\left(\frac{x-\mu}{\sigma}\right) \cdot \frac{1}{\sigma} = \frac{1}{\sqrt{2\pi} \cdot \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

(4) Given $P(Z \in [-2, 2]) = 95\%$ (approximately):

$$P(X \in [\mu - 2\sigma, \mu + 2\sigma]) = P(\mu - 2\sigma \leq \mu + \sigma Z \leq \mu + 2\sigma) = P(-2 \leq Z \leq 2) = 95\%$$

Problem 5 The PMFs/PDFs are presented in a table:

Distribution	**PMF or PDF**
Negative Binomial	$p(k) = \binom{k+r-1}{k} p^r (1-p)^k$, $k = 0, 1, \ldots$
Hypergeometric	$p(k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$, $k = \max(0, n - N + K), \ldots, \min(K, n)$
Zipf	$p(k) = \frac{k^{-s}}{\sum_{m=1}^{N} m^{-s}}$, $k = 1, \ldots, N$
Chi-square	$f(x) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2}$, $x > 0$
Student's t	$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$
Cauchy	$f(x) = \frac{1}{\pi\gamma\left[1+\left(\frac{x-x_0}{\gamma}\right)^2\right]}$
Gamma	$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$, $x > 0$
Beta	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$, $0 < x < 1$
Weibull	$f(x) = \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}$, $x \geq 0$
Gumbel	$f(x) = \frac{1}{\beta} e^{-(z+e^{-z})}$, $z = \frac{x-\mu}{\beta}$
Pareto	$f(x) = \frac{\alpha x_m^\alpha}{x^{\alpha+1}}$, $x \geq x_m$