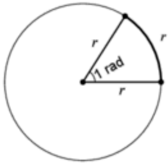
Circumference, Arc Length, and Radians

- The circumference C of a circle with a radius r and diameter $d = 2r$ is found with the following equation:

$$C = 2\pi r \quad \text{or} \quad C = \pi d$$

- For an angle measuring 1 **radian**, the length of the intercepted arc is equal to the **radius**, by definition.



The number of radians of angle measure is defined as the number of radiuses in intercepted arc length, so if we know how many radians a central angle measures, we know how many radiuses the intercepted arc measures, and if we know how many radiuses long an arc measures, we know the measure in radians of the central angle that intercepts that arc.

- There are 360° in a circle, and there are 2π radians in a circle, so we can use the following proportion when we want to convert between the degrees and radians:

$$\frac{\text{degrees}}{\text{radians}} = \frac{360^\circ}{2\pi\text{ rad}} \qquad \text{or, equivalently} \qquad \frac{\text{degrees}}{\text{radians}} = \frac{180^\circ}{\pi\text{ rad}}$$

This gives: $$\text{degrees} = \text{radians}\left(\frac{180}{\pi}\right) \qquad \text{and} \qquad \text{radians} = \text{degrees}\left(\frac{\pi}{180}\right)$$

- ***Radians* of angle produce *radiuses* of arc length**. Therefore, the formula for the length L of an arc of a circle with radius r intercepted by a central angle θ (measured in radians) is given by

$$L = r\theta$$

If you are given a central angle measure in radians, **do not convert this angle to degrees to calculate the intercepted arc length**.

- If you are given the measurement of the central angle in degrees, you can calculate the intercepted arc length by converting the angle from degrees to radians or you can find the arc's fraction of the whole circumference by using a proportion.

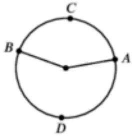
Since there are 360° in a circle, the fraction of the circle that is marked by a central angle measuring x° is $\frac{x}{360}$. The ratio of the arc length L to the whole circumference C is equal to the ratio of the central angle to 360°.

$$\frac{L}{C} = \frac{x}{360} \qquad \text{solving for } L \qquad L = \frac{x}{360}C$$

If you are not given the circumference, but you do know the radius, you can use the circumference formula and substitute $2\pi r$ for C to end up with the formula

$$L = \frac{x}{360}(2\pi r)$$

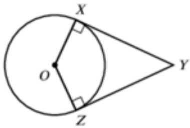
- The measure of an arc between points A and B on a circle can be denoted as $\widehat{AB}$. If there is a third point C between the points A and B, the arc may be represented as $\widehat{ACB}$. In the figure below, you may choose to refer to **minor** (smaller) **arc** $\widehat{ACB}$ in order to distinguish it from **major** (larger) **arc** $\widehat{ADB}$, which is the **complementary** arc.



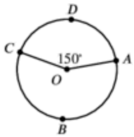
If there are only two points, and neither major nor minor is specified, assume the reference is to the minor arc.

- If you are told that lines are **tangent** to a circle, then you should note immediately that those lines **intersect the circle at exactly one point**, and a **radius drawn from the center of the circle to the point of tangency will intersect the tangent line at a right angle**.

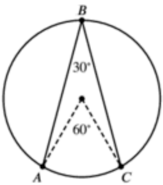
For example, in the figure below, lines $\overline{XY}$ and $\overline{ZY}$ are tangent to the circle.



- An arc can be specified by the corresponding central angle (in degrees or radians). For example, in the figure below, we can say that the arc $\widehat{ADC}$ measures 150° because the central angle forming that intercepted arc measures 150°.



- An **inscribed angle** is an angle with its vertex on the circumference of the circle. **The measure of an inscribed angle is half of the measure of the central angle of its intercepted arc**.



Area of a Circle

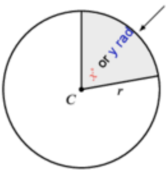
- The formula for the area A of a circle of radius r is the following:

$$A = \pi r^2$$

- The area of a sector (pie-slice-shaped portion) of a circle marked by a central angle θ is proportional to the ratio of the central angle to the angle measure of a full circle (360° or 2π radians).

$$\frac{A_{\text{sector}}}{A_{\text{circle}}} = \frac{\theta_{\text{degrees}}}{360^\circ} \quad \text{or} \quad \frac{A_{\text{sector}}}{A_{\text{circle}}} = \frac{\theta_{\text{radians}}}{2\pi}$$

Therefore, the area of a sector marked by a central angle measuring x° or y radians is found with the following equations:



$$\frac{A_{\text{sector}}}{A_{\text{circle}}} = \frac{x^\circ}{360^\circ} \quad \text{or} \quad \frac{A_{\text{sector}}}{A_{\text{circle}}} = \frac{y}{2\pi}$$

$$A_{sector} = \frac{x^\circ}{360}(\pi r^2) \quad \text{or} \quad A_{sector} = \frac{y \text{ rad}}{2\pi}(\pi r^2)$$

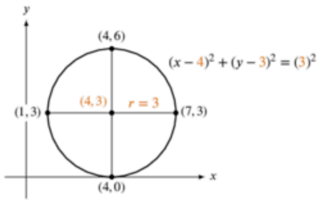
The Circle Equation

- The equation for all points (x, y) on a circle **centered at the origin** with radius r is derived from the Pythagorean Theorem:

$$x^2 + y^2 = r^2$$

For any circle with radius r, **centered at a point (h, k)**, the equation for all points (x, y) on the circle is slightly modified to reflect the shifted center point:

$$(x - h)^2 + (y - k)^2 = r^2$$



All points strictly inside of a circle conform to the inequality $(x - h)^2 + (y - k)^2 < r^2$. All points strictly outside of a circle conform to the inequality $(x - h)^2 + (y - k)^2 > r^2$.

In order to write a circle equation in the form $ax^2 + bx + ay^2 + cy + d = e$ in Standard Form, $(x-h)^2 + (y-k)^2 = r^2$, follow these steps:

1. Isolate all constant terms on the opposite side of the equation from the x- and y-terms.
2. Divide both sides of the equation by a, the coefficient of the x^2 and y^2 terms, making sure to distribute the division across all terms. The x^2 and y^2 coefficients will always match for circle equations, and most often on the test will be 1, so this step will often be unnecessary.
3. Complete the square for both x- and y-variable polynomial expressions.

   If the x-coefficient is p and the y-coefficient is q, you should rewrite the x terms as $\left(x+\frac{p}{2}\right)^2$, rewrite the y terms as $\left(y+\frac{q}{2}\right)^2$, and add $\left(\frac{p}{2}\right)^2$ and $\left(\frac{q}{2}\right)^2$ to the other side of the equation.
4. The equation is now in Standard Form for circles, and you can therefore identify the center point and radius.

$2x^2 + 12x + 2y^2 + 16y - 98 = 52$

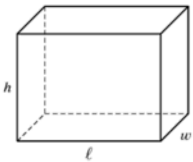
$2x^2 + 12x + 2y^2 + 16y = 150$	Isolate constants on the right side of the equation
$x^2 + 6x + y^2 + 8y = 75$	Divide by the common x^2 and y^2 coefficient
$(x+3)^2 + y^2 + 8y = 75 + 9$	Complete the square for the x-terms
$(x+3)^2 + (y+4)^2 = 75 + 9 + 16$	Complete the square for the y-terms
$(x+3)^2 + (y+4)^2 = 100$	
$(x+3)^2 + (y+4)^2 = 10^2$	

Volume

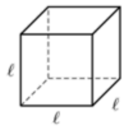
The **volume of a solid or three-dimensional figure is the amount of space that the object occupies** and is usually measured in units^3 (though the volume of fluids like water and oil might be measured in units like ounces or liters).

A **right rectangular prism** is a box (all faces meet at right angles) with 6 rectangular faces and dimensions of length ℓ, width w, and height h.

A **cube** is a special case of rectangular prism where all three dimensions are equal (there is one common side length ℓ) and therefore, every face is a square.



Rectangular Prism



Cube

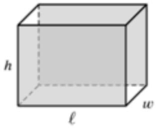
For right rectangular prisms, the volume formula is given to you in the info box at the beginning of test sections.

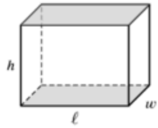
$$V = \ell w h$$

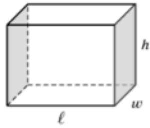
The volume of a cube with side length s is simplified because all three dimensions are equal:

$$V = s \cdot s \cdot s \qquad \text{or} \qquad V = s^3$$

- To find the surface area of a right rectangular prism, add up the areas of all 6 faces of the object. As shown in the figures below, there are two faces that are rectangles measuring ℓ by h, two faces that are rectangles measuring ℓ by w, and two faces measuring h by w.





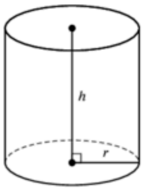


$$S = 2(\ell h + \ell w + wh)$$

For cubes with sides of length ℓ:

$$S = 6\ell^2$$

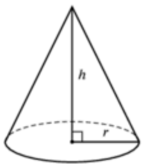
- A **right circular cylinder** is a three-dimensional figure with a circular base of radius r and height of h.



The volume of a cylinder is given to you in the info box at the beginning of math sections on the test, and it's the area of the circular base times the height of the object.

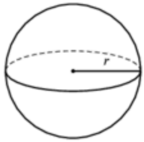
$$V_{cyl} = \pi r^2 h$$

- Right circular cones have a circular base with radius r, but unlike cylinders, cones taper to a point as they reach their height h.



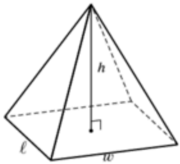
$$V = \frac{1}{3}\pi r^2 h$$

- A **sphere** is the shape formed by all points in three dimensions that are the same distance, r (a radius), away from the center point.



$$V = \frac{4}{3}\pi r^3$$

- A **right rectangular pyramid** has a rectangular base of dimensions ℓ by w and four triangular faces that meet at a point directly above the center of the base at a height of h.



$$V = \frac{1}{3}\ell w h$$

- The **density D of an object is equal to the mass of the object m divided by its volume V:**

$$D = \frac{m}{V}$$