Percentages

Percentages

- In any situation involving percentages, there are three fundamental components:
  1. The reference, base, total, starting, or original amount of something
  2. The relative portion or share of the reference amount, which will be given or requested as a percentage
  3. The actual value of the relative portion or share, which will be in the same units as the reference value
- You can write basic percentage equations in either of these forms:

$$\frac{\text{portion of the total}}{\text{total}} = \text{relative amount} \quad \text{or} \quad \text{portion of total} = (\text{relative amount})(\text{total})$$

- You can always orient yourself in percentage problems by putting all the provided information into the form, "A is $P\%$ of B," which translates to the second equation form above written as

$$A = pB$$

  where the lowercase p is the decimal representation of P.
- Convert from decimal to percentage by multiplying the decimal by 100; this is easily accomplished by moving the decimal point two places to the right, adding zeros as needed.

  Convert from percentage to decimal by dividing the percentage by 100, easily done by moving the decimal point two places to the left, adding zeros as needed.

$$32\% \text{ is } 0.32 \text{ and } 1.24\% \text{ is } 0.0124$$

Percent Increase / Decrease

- When a value is **decreased** by a percentage, you need to multiply the value by the complementary decimal value of the percentage, which is **1 MINUS the decrease in decimal form**, to find the remaining amount. For example, if a problem tells you that something **loses, decreases by, declines by, is reduced by, is discounted by, or shrinks by 20%, you will need to multiply the number by 0.8, not 0.2, because the new value is $100\% - 20\% = 80\%$ of the original value**.
- When a value is **increased** by a percentage, you need to multiply the value by the decimal value of the percentage, which is **1 PLUS the increase in decimal form**, to get the total amount after the increase. For example, if a problem tells you that something **grows or increases by 20%, you will need to multiply the number by 1.2 because the new value is $100\% + 20\% = 120\%$ of the original value**.
- Percent More $\neq$ Percent Less

  It is important to note that 150 is 50% *more* than 100 because $1.5(100) = 150$, but 100 is obviously NOT 50% *less* than 150 because 50% (or half) of 150 is 75.

  Write your equations based on what the question tells you to write. Look for the words "is" or "equals" to tell you on which side of the equation to write certain terms. If x *is* 80% of y, write the equation $x = 0.8y$.

  It is INCORRECT to infer that if x is 20% less than y, then y is 20% more than x. Given the statement "x is 80% of y," you should NOT write $y = 1.2x$, but rather $x = 0.8y$, which means $y = \frac{x}{0.8}$ or $y = 1.25x$.

- You can calculate percent increases and decreases using the same equation $A = pB$, where A is the new amount and B is the original amount (read as "the new amount, A, is P percent of the old amount, B," where P is the percentage equivalent of the decimal p). The decimal p that is solved for is relative to 1 (indicating what percent A is of B).

  If $p = 1.2$, there was a 20% growth from B to A (that is, A is 120% of B).

  If $p = 0.85$, there was a 15% decrease from B to A (that is, A is only 85% of B, so B decreased by 15%).

- Percent Change Formula:

$$\text{Percent Change} = \frac{\text{New Value} - \text{Original Value}}{\text{Original Value}}$$

  The decimal is the percent change from the original value, and the sign (positive or negative) tells you whether the change is an increase or decrease.

- If a value changes by a certain percentage and then changes again by a certain percentage, you need to *multiply* the original value by the percentages, NOT add the percentages and then apply the result to the original value, because the reference value changes during the sequence.

  For example, if an item cost \$100 initially, but the cost is reduced by 10% (which changes the price to \$90), and then you use a 10% off coupon, you are taking 10% off of the new listed price of \$90, not the original price. Therefore, the price you pay is 90% of 90% of \$100 (you do NOT pay 80% of the original price):

$$\text{price} = 0.9(0.9)(100) = .9(90) = 81$$

Mixtures & Concentrations

- The percentage equations in solution and mixture problems are just linear equations with decimals.

  The amount of a substance in a solution/mixture is equal to the percentage of the solution that is made up of that substance times the total amount of the solution/mixture. For example, if there are A kilograms of a saline solution (a solution of salt in water) that is 10% salt by mass, then $0.1A$ is the amount of salt in that solution.

  When mixing solutions to form a new solution, add the amounts of a substance in the ingredients (the percentage of the solution that is made up of that substance times the amount of the solution) for the two solutions and set them equal to the amount of that substance in the resulting mixture (which, again, is the percentage of the solution that is made up of that substance times the total amount of the solution). For example, if we mix A kilograms of a 10% saline solution and B kilograms of a 20% saline solution to form a new 15% saline solution, we can write the following equation

$$0.1A + 0.2B = 0.15(A + B)$$

  because the amount of salt in the first solution is 10% of the total amount of that solution (which is A kilograms), the amount of salt in the second solution is 20% of the total amount of that solution (which is B kilograms), and the amount of salt in the new solution is 15% of the total amount of that solution (which is $A + B$ kilograms).

Exponential Relationships

Exponential Equations

- Exponential change is characterized by an initial value that is repeatedly multiplied or divided by the same amount.
- Most exponential equations you will have to write or interpret will deal with percent increase (growth) or decrease (decay) and will be of the form

$$y = a \cdot b^x$$

  where a is the initial value, b is the rate of change (the amount that the value is multiplied by over each interval), and x is the number of intervals (usually a time interval).
- In exponential equations where the initial equation's exponent is written in terms of one unit of measurement, but you are supplied with the period or interval variable or value in different units, you should use a proportion showing the relative values of the units to determine the value of the exponent when expressed in the original units.

  For example, there are four quarters in one year, so we can use the proportion $\frac{q}{t} = \frac{4}{1}$, where q is the number of quarters and t is the number of years, to convert measurements in one unit to the other.

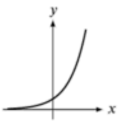
Graphs of Exponential Equations

- It is useful to know the general shapes of exponential graphs, where one side of the graph is almost horizontal and the other side is almost vertical. Plugging in test values and graphing a few points should give you a reasonable understanding of the graph for any particular exponential equation.

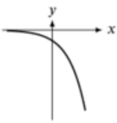
  In general for equations of the form $y = ab^{cx}$:

  - Positive values of a indicate that the graph has a positive y-intercept and will be entirely above the x-axis.
  - Negative values of a indicate that the graph has a negative y-intercept and will be entirely below the x-axis.
  - When $b \geq 1$ and c is positive AND when $0 < b < 1$ and c is negative, the graph levels off to the left side and goes to infinity (or negative infinity when a is negative) on the right side (the y-value changes slowly, then changes rapidly).

$y = 2(2)^x$ or $y = 2\left(\frac{1}{2}\right)^{-x}$



$y = -2(2)^x$ or $y = -2\left(\frac{1}{2}\right)^{-x}$

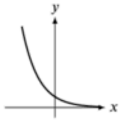


➔ When $b \geq 1$ and c is negative AND when $0 < b < 1$ and c is positive, the graph levels off to the right side and goes to infinity (or negative infinity when a is negative) on the left side (the y-value changes rapidly, then changes slowly).

$y = 2(2)^{-x}$

or

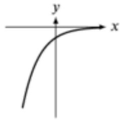
$y = 2\left(\frac{1}{2}\right)^x$



$y = -2(2)^{-x}$

or

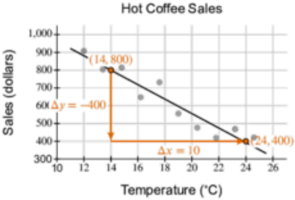
$y = -2\left(\frac{1}{2}\right)^x$



Scatterplots and Line Graphs

Scatter Plots

- Find the equation of a line of best fit for a scatterplot in the same ways that you did for any regular line. If no line of best fit is drawn, try your best to draw one in that runs roughly through the center of the cluster of points.

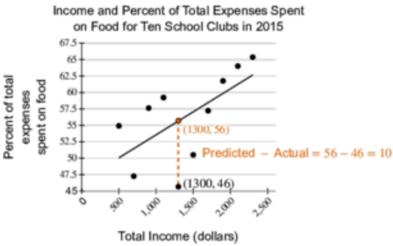


$$m \approx \frac{\Delta y}{\Delta x} = \frac{-400}{10} = -40$$

⇓ Substitute slope and point (14, 800) to solve for b:

$$y = mx + b$$
$$800 = -40(14) + b$$
$$800 - 560 = b$$
$$1{,}360 = b$$
$$\Downarrow$$
$$y = -40x + 1{,}360$$

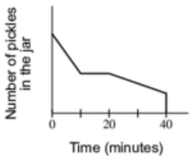
- Find the difference between actual and predicted values by seeing how far above or below the line of best fit a data point is.



- Do not make overly definite statements about what the slopes and intercepts of scatterplots tell us. They are merely models and sometimes only fit very roughly over particular intervals.

Reading Line Graphs

- Most line graph questions are simply related to the slope of the line on certain intervals. Steep lines mean something changed rapidly during an interval. Horizontal lines means something stayed the same for an interval. Vertical lines mean something changed instantaneously.
For example, Hannah eats pickles while she studies. She eats half of the pickles during the first 10 minutes of studying. After eating half of the pickles, she stops eating for the next 10 minutes. Then she gradually eats the pickles until she purposely spills all of the remaining pickles.



Functions

Evaluating Functions ("Plugging In")

- When plugging into a function $f(x)$, replace each instance of x with the value or expression that is being plugged into the function. Replace each x with a set of blank parentheses as an intermediate step if you have trouble plugging into the function.

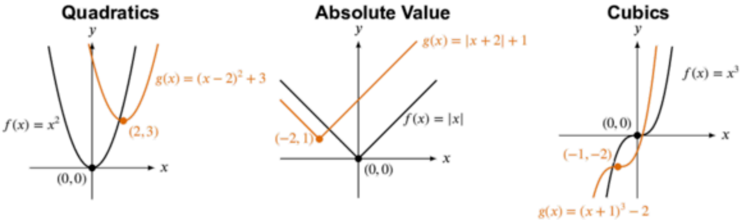
$$f(x) = x^2 - 2x + 1 \quad \Rightarrow \quad f(\quad) = (\quad)^2 - 2(\quad) + 1 \quad \Rightarrow \quad \begin{array}{l} f(-3) = (-3)^2 - 2(-3) + 1 \\ f(x+2) = (x+2)^2 - 2(x+2) + 1 \end{array}$$

- When working with composite functions, plug the input value into the inside function first and then plug the resulting value into the outside function.

$$\begin{array}{l} g(x) = 4x + 2 \\ h(x) = 3 - g(x) \end{array} \quad \Rightarrow \quad g(0) = 4(0) + 2 = 2 \quad \Rightarrow \quad h(0) = 3 - g(0) = 3 - 2 = 1$$

Function Shifting and Unfamiliar Graphs

- The graph of a function $g(x) = f(x-h) + k$ is shifted h units horizontally and k units vertically from the graph of $f(x)$.



- If you are given an unfamiliar graph type and are asked to find the corresponding function, plug identifiable points, such as the y-intercept or x-intercept, into each of the answer choices, eliminating any that don't fit all of the tested points.

Statistics

Data Sets: Mean, Median, Mode, and Range

- When trying to find the key properties of a data set, you may find it useful to rewrite the data set in order from least to greatest to help prevent mistakes.
- The **mean** is the average of the values in a data set. The mean is found by dividing the sum of the values in the data set by the size of the data set.

$$\text{Mean} = \frac{\text{Sum of Data Set}}{\text{Size of Data Set}}$$

- The **median** is located in the middle of the data set.

  For an odd-sized data set, the position of the median is found by adding 1 to the size of the data set and then dividing by 2. For example, the median of a data set with 49 values is found in position 25 (the 25th value in the data set) because $\frac{49+1}{2} = \frac{50}{2} = 25$.

  For an even-sized data set, the median is found by averaging the two middle values. The positions of these two values are found by dividing the size of the data set by 2; the result gives the position of the first middle value, and the other middle value is found at the next position in the data set. For example, the median of a data set with 100 values is found by averaging the values of the 50th and 51st values in the data set because $\frac{100}{2} = 50$.
- The **mode** of a data set is the value that appears most frequently in a data set.
- The **range** of a data set is the difference between the largest and smallest value in a data set. Subtract the smallest value from the largest value to find the range.

Process	**Set A:** $\{4, 2, 8, 10, 1, 9, 8\}$	**Set B:** $\{2, 0, 0, 4, 5, 5\}$
Increasing Order	$\{1, 2, 4, 8, 8, 9, 10\}$	$\{0, 0, 2, 4, 5, 5\}$
Mean	$\text{Mean}_A = \frac{1+2+4+2(8)+9+10}{7} = \frac{42}{7} = 6$	$\text{Mean}_B = \frac{2(0)+2+4+2(5)}{6} = \frac{16}{6} = \frac{8}{3}$
Median	$A = \{1, 2, 4, 8, 8, 9, 10\}$	$B = \{0, 0, 2, 4, 5, 5\}$ $\text{Median}_B = \frac{2+4}{2} = \frac{6}{2} = 3$
Mode	$\text{Mode}_A = 8$ This value appears twice, and no other value is repeated	No value appears more often than any other: there is **no mode**.
Range	$\text{Range}_A = 10 - 1 = 9$	$\text{Range}_B = 5 - 0 = 5$

Frequency Tables, Histograms, and Standard Deviation

➔ Frequency tables and histograms are used to show how many times particular values occur in a data set. From these representations, we are able to calculate all of the key properties of data sets. To find the mean, we need to find the sum of the data, which can be determined by multiplying each value in the data set by its frequency (or the height of its bar) and summing the results, then dividing by the number of samples. The median is found by counting the number of elements until you reach the position of the median (in the middle of the set).

Ages of 200 People Enrolled in a Hot Yoga Studio

Age	Frequency	
18	34	People 1–34
19	21	People 35–55
23	37	People 56–92
25	38	People 93–130
30	46	
45	24	

$$\text{Mean Age} = \frac{34(18) + 21(19) + 37(23) + 38(25) + 46(30) + 24(45)}{200}$$

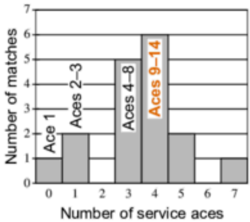
$$\text{Mean Age} = \frac{5{,}272}{200}$$

$$\text{Mean Age} = 26.36$$

There are 200 values in the set, so the median is the average of the 100th and 101st values in the set, both of which are 25 years old.

$$\text{Median Age } = 25$$

Number of Service Aces by a Volleyball Team in 17 Matches



$$\text{Mean Aces} = \frac{1(0) + 2(1) + 5(3) + 6(4) + 2(5) + 1(7)}{17}$$

$$\text{Mean Aces} = \frac{58}{17}$$

$$\text{Mean Aces} \approx 3.41$$

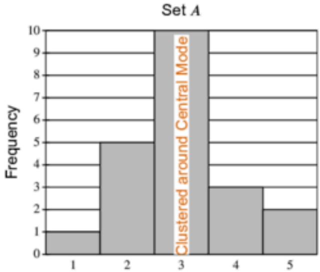
There are 17 values in the set, so the median is the 9th value in the set, which is 4 aces.

$$\text{Median Aces } = 4$$

➔ **Standard deviation** is a measure of how closely clustered the values in a data set are (how close to the mean of the data most of the values are). Tightly clustered data sets will have a lower standard deviation than will data sets that are more spread out.

Set A	
Value	Frequency
1	1
2	5
3	**10**
4	3
5	2

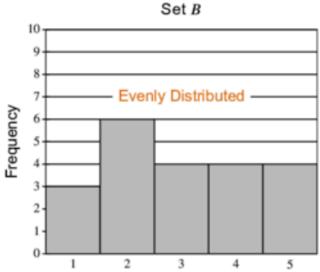
Clustered around Central Mode



Lower Standard Deviation

Set B	
Value	Frequency
1	3
2	6
3	4
4	4
5	4

Evenly Distributed

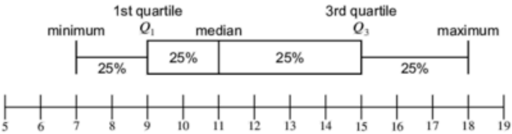


Higher Standard Deviation

Box Plots

- A box plot is formed by drawing short vertical lines for the median and quartiles Q_1 and Q_3 and then connecting the tops and bottoms of those lines to form a box. The outermost vertical lines mark the maximum and minimum values in the set. About 25% of the data set will be found between each vertical line on the box plot (each section contains roughly the same amount of data).

  For the box plot in the figure below, the median is 11; the first quartile is 9, and the third quartile is 15; the minimum is 7, and the maximum is 18.



Survey Design / Interpreting Results

- In order to draw valid conclusions from surveys, the samples must be sufficiently large (so far there has never been a sample that was too small on any released tests because, regardless of the population size, even seemingly small sample sizes are fairly accurate), and **most importantly, truly random**. If the subjects in a survey share some trait (other than just being members of the larger population), then the survey's results are restricted to just the population that shares that trait and cannot be applied to the larger population to which those subjects belong.
- The **margin of error** accounts for the potential difference between the true value for an entire population and the value found based on a survey sample. The true value for the whole population is most likely (but not definitely) found inside the margin of error around the value found for the survey. A larger sample size leads to a smaller margin of error.

  An ecologist selected a random sample of 50 beavers from a river and found that the mean weight of the beavers in the sample was 42 pounds (lbs), with an associated margin of error of 3.1 lbs. Therefore, any weight between $42 - 3.1 = 38.9$ lbs and $42 + 3.1 = 45.1$ lbs is a plausible value for the mean weight of all the beavers in the river.
- Do not make overly definitive claims based on a survey. The results only apply to the population that shares the traits common to those sampled. A cause-and-effect relationship is highly unlikely unless you are specifically told that all other variables have been controlled for. You are a lot safer merely making a claim just that there is a correlation between two things (rather than a cause-and-effect relationship).

Unit Conversions

Simple Conversions with Proportions

- When converting from one unit to another unit based on a direct conversion rate, you can simply use a proportion to solve the problem.

  EXAMPLE:
  Convert 3 inches to centimeters ($1\text{ inch} = 2.54\text{ cm}$):

$$\frac{\ell_{\text{cm}}}{\ell_{\text{in}}} = \frac{2.54\text{ cm}}{1\text{ in}} \quad \Rightarrow \quad \frac{\ell_{\text{cm}}}{3\text{ in}} = \frac{2.54\text{ cm}}{1\text{ in}} \quad \Rightarrow \quad \ell_{\text{cm}} = \left(\frac{2.54\text{ cm}}{1\cancel{\text{ in}}}\right)(3\cancel{\text{ in}}) \quad \Rightarrow \quad \ell_{\text{cm}} = 7.62\text{ cm}$$

Factor-Label Method

- Use the Factor-Label method when a direct conversion from one unit to another is not available. String together unit conversion factors in order to cancel one unit at a time until the desired units are achieved.

  EXAMPLE:
  Convert 76 inches to meters ($1\text{ inch} = 2.54\text{ cm}$ and $1\text{ m} = 100\text{ cm}$):

$$76\cancel{\text{ in}}\left(\frac{2.54\cancel{\text{ cm}}}{1\cancel{\text{ in}}}\right)\left(\frac{1\text{ m}}{100\cancel{\text{ cm}}}\right) = 1.93\text{ m}$$

- Identify a **starting point** for word problems with unit conversions by picking out information that tells us about the **end goals**. In order to determine the steps needed after the starting point is identified, try to cancel or convert unwanted units, one-by-one, based on additional information in the problem.

The Distance-Speed-Time Equation

- The distance-speed-time equation is

$$d = st$$

  where d is distance, s is speed, and t is time.

Angles, Triangles, and Trigonometry

Angles

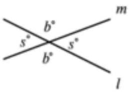
- Angles are formed by the intersection of two lines; the point at which the two lines meet is called the **vertex** (plural, **vertices**).
- Angles are most commonly measured in **degrees**, which is denoted by the degree symbol (°).
- A 90°angle is a **right angle** (the lines forming the angle are perpendicular). Right angles are often marked with a small square instead of an arc and angle measurement.

Any angles that together form a right angle sum to 90°. When two angles form a 90° angle they are called **complementary angles**.

- A 180° angle results in a straight line.

Any angles that together form a straight line sum to 180°. When two angles form a 180° angle they are called **supplementary angles**.

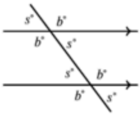
- Angles that measure more than 0° and less than 90° are called **acute angles**. Angles that measure more than 90° and less than 180° are called **obtuse angles**. Angles greater than 180° are called **reflex** angles.
- Whenever two lines intersect, four angles are formed, and the angles opposite each other, called **Vertical Angles**, are equal.

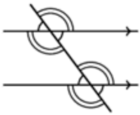


As previously explained, angles that form a line sum to $180°$. Therefore, $s + b = 180$.

- To indicate congruence (equality of measure) for sides and angles, those features can be marked with dashes or arcs, called hatch (or hash or tick) marks. Sides marked with the same number of hatch marks are congruent, and angles marked with the same number of arcs are congruent.

  Parallel lines can be marked by arrow heads. Lines marked with the same number of arrow heads are parallel.
- When a diagonal (non-perpendicular) line runs across two parallel lines, eight angles are formed. The four larger (obtuse) angles are all equal to each other. Similarly, the four smaller (acute) angles are also equal to each other.

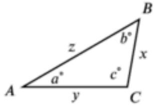




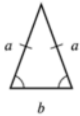
Triangles and Other Polygons

- **The interior angles of any triangle sum to 180°**. The length of each side is positively correlated with the size of the angle opposite that side, so the **largest side of a triangle is always across from the largest angle of the triangle, and the smallest side is across from the smallest angle**.

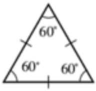
  In the triangle below, $a + b + c = 180$ because the sum of any triangle's interior angles is 180°. Since $\angle C$ is the largest angle, z is the longest side. Similarly, since $\angle A$ is the smallest angle, x is the shortest side.



- **In isosceles triangles, two of the angles are equal to each other, and the sides across from those angles are therefore also equal to each other.**



- **In equilateral triangles, all of the sides are equal to each other, and therefore all of the angles are also equal**. Because all three of the angles are equal and they must sum to 180°, each angle is $\frac{180^\circ}{3} = 60^\circ$.

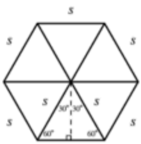


- For a polygon with n sides and angles, the sum of the interior angles can be found using the following formula, which is very rarely needed:

$$\text{Sum of Interior Angles} = 180(n - 2)$$

- In a **regular polygon**, all sides are equal, and all angles are equal. The measure of any angle is equal to the sum of the interior angles divided by the number of angles.

- **Regular hexagons** (six-sided polygons) can be split into 6 equilateral triangles if we draw a line from the center of the figure to each of the six vertices.



Similar Triangles

- Two triangles are called **Similar Triangles** when they have the same interior angles. Put another way, each angle in one triangle has a corresponding match in the other triangle.

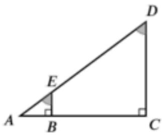
In the figure above, $\triangle ABC$ is similar to $\triangle DEF$ because all of the angles in $\triangle ABC$ have a matching angle in $\triangle DEF$. Notice that the triangles are different sizes, however. **In similar triangles, all of the angles have a match, but the sides are not *necessarily* equal. However, sides across from matching angles are always in the same proportion to each other.**

In the figure shown above, $\overline{AB}$ corresponds to $\overline{DE}$; $\overline{BC}$ corresponds to $\overline{EF}$; and $\overline{AC}$ corresponds to $\overline{DF}$. In these particular triangles, the sides in $\triangle ABC$ are all twice the length of the corresponding sides in $\triangle DEF$.

- **If we know (or can determine) that there are two angles that are the same in two triangles, the third angle must also be the same** because the three angles in each triangle must sum to the same value (180°). This, in turn, means that **the two triangles are similar triangles**.

- Similar triangles are commonly depicted as one triangle inside of another, where you can show that the triangles must be similar because all the angles in both triangles will have a match.

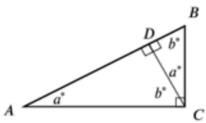
In the figure below, $\triangle ACD$ is similar to $\triangle ABE$ because they both contain $\angle A$ and a right angle, which means the third angles in the two triangles must also be equal.



- You can use part-to-whole or part-to-part ratios to solve for lengths in divided triangles. When a **triangle is divided by a line parallel to one side, the other two sides are divided proportionally**; this is the **Triangle Proportionality Theorem**.

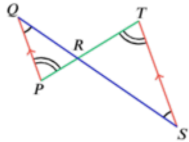
In the triangle above, the Triangle Proportionality Theorem tells us that $\frac{AE}{ED} = \frac{AB}{BC}$.

- Right triangles can be divided with a single line that creates three similar triangles. In the figure below, a line is drawn from the right angle vertex C that intersects the hypotenuse at a right angle at point D. The larger right triangle is split into two smaller right triangles, ΔADC and ΔCDB, both of which are similar to the original triangle ΔACB and thus to each other as well.



- Similar triangles arise in arrangements such as that shown in the figure below, with the two similar triangles touching at one vertex (where there is a vertical angle) and each having a side that is parallel to a side in the other triangle.

  In the figure below, the angles meeting at point R are vertical angles (and thus are equal to each other), and side $\overline{PQ}$ is parallel to side $\overline{ST}$.



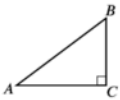
$$\Delta PQR \sim \Delta TSR$$
and
$$PQ : QR : RP = TS : SR : RT$$

Trigonometry

- In a right triangle, the side that is across from the right angle is called the **hypotenuse** ($\overline{AB}$ in the triangle below).

  The side across from an acute angle is called the **opposite** side. In the triangle below, side $\overline{BC}$ is the opposite side for $\angle A$, and side $\overline{AC}$ is the opposite side for $\angle B$.

  The side that forms an angle with the hypotenuse is called the **adjacent** side of that angle. In the triangle below, side $\overline{AC}$ is the adjacent side for $\angle A$, and side $\overline{BC}$ is the adjacent side for $\angle B$.



$$\sin A = \frac{BC}{AB} \quad \sin B = \frac{AC}{AB}$$
$$\cos A = \frac{AC}{AB} \quad \cos B = \frac{BC}{AB}$$
$$\tan A = \frac{BC}{AC} \quad \tan B = \frac{AC}{BC}$$

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

- The trig ratios can be remembered using the mnemonic **SOH-CAH-TOA**, which distills the information that **S**ine is **O**pposite over **H**ypotneuse, **C**osine is **A**djacent over **H**ypotenuse, and **T**angent is **O**pposite over **A**djacent.
- Trigonometric functions' values are derived from the side ratios **in a right triangle**, and their **values for a given angle don't change** if the angle appears in a figure that is not a right triangle; the **sine of an angle is the sine of that angle regardless of what type of figure** includes that angle. However, you can **only calculate these functions from a figure when the angle is part of a right triangle**.
- The sine and cosine of complementary angles have the following relationship:

$$\sin(x^\circ) = \cos(90^\circ - x^\circ) \quad \text{and} \quad \cos(x^\circ) = \sin(90^\circ - x^\circ)$$

  This is useful because the acute angles in a right triangle sum to 90°.

Perimeter & Area

- The **perimeter** of a polygon is the sum of the lengths of the outside edges.
- **Area** is a measure of the space that a shape covers on a plane. Area is measured in units of length squared.
- The area of a rectangle of length ℓ and width w is given by

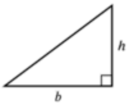
$$A = \ell w$$

The area of a square of side length s is given by

$$A = s^2$$

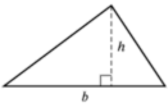
- For a triangle with a base length b and height h, the area A is found with the following formula:

$$A = \frac{1}{2}bh$$



- The height of a triangle is always perpendicular to the base.

To show the height for acute triangles (all angles smaller than 90°), you usually have to draw in a line that is perpendicular to the base and that divides the triangle into two right triangles.



For obtuse triangles (one angle is larger than 90°), depending on the orientation, you may need to draw the perpendicular line for the height outside of the triangle as in the first picture below. Alternatively, you can always rotate the triangle (as in the second picture below) so that the perpendicular height line goes into the largest angle.

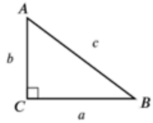
- If you need to find the area of other polygons, divide them into triangles and rectangles so that you can sum the areas of the individual sections.

Pythagorean Theorem & Special Right Triangles

- If you know two of the side lengths of a right triangle, you can use the **Pythagorean Theorem** to find the length of the third side.

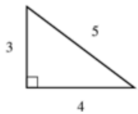
  The Pythagorean Theorem states the following for a right triangle with legs measuring a units and b units and hypotenuse measuring c units:

$$a^2 + b^2 = c^2$$



- There are some special right triangles in which all three sides of the triangle are whole numbers (or are in whole number ratios with each other). These are called **Pythagorean Triples**.
- The most common Pythagorean Triple is the $\mathbf{3:4:5}$ **triangle**.

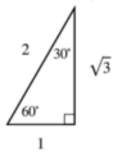
  In a right triangle with legs of length 3 and 4, the hypotenuse has length 5.



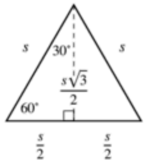
$$\begin{aligned} a^2 + b^2 &= c^2 \\ 3^2 + 4^2 &= 5^2 \\ 9 + 16 &= 25 \\ 25 &= 25 \end{aligned}$$

  The actual side lengths can be any set of values that conform to the $3:4:5$ ratio.

- Other Pythagorean Triples include $\mathbf{5:12:13}$ triangles (occasionally seen), and the far rarer $\mathbf{7:24:25}$ triangles, $\mathbf{8:15:17}$ triangles, and $\mathbf{20:21:29}$ triangles.
- The test makes extensive use of **angle-based special right triangles** because their sides are in easy-to-write-and-remember ratios to each other.
- The first important angle-based special right triangle is a **30-60-90 right triangle**, which has a 30°, 60°, and 90° angle. Its sides, from shortest to longest, are in a ratio of $1:\sqrt{3}:2$.

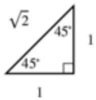


➔ **One of the most common ways that 30-60-90 triangles are hidden is in equilateral triangles**. If you divide an equilateral triangle with side lengths s in half by drawing a line from any vertex to its opposite side and intersecting that opposite side at a right angle, the triangle is divided into two 30-60-90 triangles where the short legs of these triangles have length $\frac{s}{2}$ and the height of the triangles is $\frac{s\sqrt{3}}{2}$.

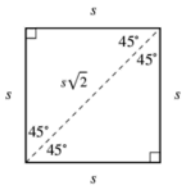


Note that this can be useful when dealing with regular hexagons, which can be divided into 6 equilateral triangles, which can then be divided into 30-60-90 triangles.

➔ The second important angle-based special right triangle is a **45-45-90 right triangle**. Since **two angles are the same, these triangles are isosceles, meaning that both legs are the same length**. The sides are in a ratio of $1:1:\sqrt{2}$.



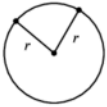
➔ **One of the most common ways that 45-45-90 triangles are hidden is in squares**. If you divide a square with side lengths s in half diagonally from one vertex to its opposite, the square is divided into two 45-45-90 triangles where the legs have length s and the hypotenuse has length $s\sqrt{2}$.



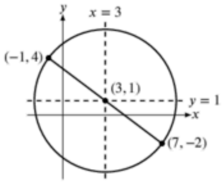
Circles & Volume

Circles & Angles

- A circle is a shape formed by **all of the points that are the same distance away from one central point** (the center of the circle). The **radius**, marked r in the figure below, is the distance from any point on the circle to the center point.



- A **diameter** is a line from one point on the arc to its opposite point on the arc that goes directly through the center of the circle and whose length, d, is equal to twice the radius: $d = 2r$.
- The center point of the circle is the midpoint between the endpoints of any diameter. This center point's x- and y-coordinates can be found by averaging the x- and y-coordinates, respectively, of the endpoints of any diameter.



$$x_{center} = \frac{(-1)+7}{2} \qquad y_{center} = \frac{4+(-2)}{2}$$

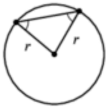
$$x_{center} = \frac{6}{2} \qquad y_{center} = \frac{2}{2}$$

$$x_{center} = 3 \qquad y_{center} = 1$$

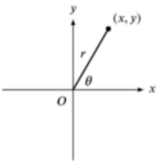
Center $= (x_{center}, y_{center})$

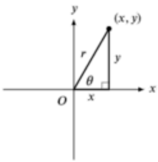
Center $= (3, 1)$

- If a triangle is formed with one vertex at the center of the circle and the other two vertices at points on the circle, then two sides of the triangle are equal to the radius, and thus the triangle is isosceles.

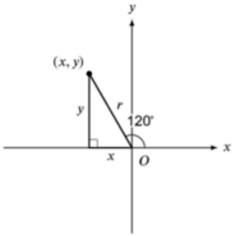


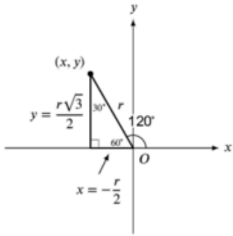
- When we graph a line segment of length r having one end at the origin, extending to a point (x, y), and lying at an angle θ measured relative to the positive x-axis, we can form a triangle as shown below. The horizontal leg has length x, and the vertical leg has length y. The values of x and y can be positive or negative, while r is always positive.





When determining absolute or relative side lengths for such a triangle, be sure to assign the correct sign. In the example below, the short leg lies along the negative x-axis, so its length will be negative. It is essential to have the proper signs for the side lengths when constructing trigonometric ratios for angles represented on the coordinate plane.

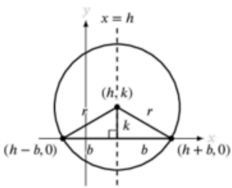


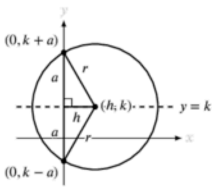


The angle ratios of a right triangle drawn on the coordinate plane are still calculated the same way (remember SOH-CAH-TOA).

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$$

Pairs of points on the circle that have the same y-coordinate will be equidistant from the vertical axis of symmetry, and the distance between such points and the axis of symmetry is simply the difference in the points' x-coordinates and the axis of symmetry's (and thus the center point's) x-coordinate, as seen below, left. With respect to the horizontal axis of symmetry, the same principle applies for pairs of points that have the same x-coordinate, as seen on the right below.





If we're not given the coordinates of the center point, but we have the coordinates of two points on the circle with either the same x- or y-value, **average the x- or y-coordinates of the two points on the circle to find the corresponding coordinate of the center point**.