Percentages

Percentages

- In any situation involving percentages, there are three fundamental components:

  1. The reference, base, total, starting, or original amount of something
  2. The relative portion or share of the reference amount, which will be given or requested as a percentage
  3. The actual value of the relative portion or share, which will be in the same units as the reference value

- You can write basic percentage equations in either of these forms:

$$\frac{\text{portion of the total}}{\text{total}} = \text{relative amount} \quad \text{or} \quad \text{portion of total} = (\text{relative amount})(\text{total})$$

- You can always orient yourself in percentage problems by putting all the provided information into the form, "A is $P\%$ of B," which translates to the second equation form above written as

$$A = pB$$

  where the lowercase p is the decimal representation of P.

- Convert from decimal to percentage by multiplying the decimal by 100; this is easily accomplished by moving the decimal point two places to the right, adding zeros as needed.

  Convert from percentage to decimal by dividing the percentage by 100, easily done by moving the decimal point two places to the left, adding zeros as needed.

$$32\% \text{ is } 0.32 \text{ and } 1.24\% \text{ is } 0.0124$$

Percent Increase / Decrease

- When a value is **decreased** by a percentage, you need to multiply the value by the complementary decimal value of the percentage, which is **1 MINUS the decrease in decimal form**, to find the remaining amount. For example, if a problem tells you that something **loses, decreases by, declines by, is reduced by, is discounted by, or shrinks by 20%, you will need to multiply the number by 0.8, not 0.2, because the new value is $100\% - 20\% = 80\%$ of the original value**.

- When a value is **increased** by a percentage, you need to multiply the value by the decimal value of the percentage, which is **1 PLUS the increase in decimal form**, to get the total amount after the increase. For example, if a problem tells you that something **grows or increases by 20%, you will need to multiply the number by 1.2 because the new value is $100\% + 20\% = 120\%$ of the original value**.

- Percent More ≠ Percent Less
  It is important to note that 150 is 50% *more* than 100 because $1.5(100) = 150$, but 100 is obviously NOT 50% *less* than 150 because 50% (or half) of 150 is 75.

  Write your equations based on what the question tells you to write. Look for the words "is" or "equals" to tell you on which side of the equation to write certain terms. If x *is* 80% of y, write the equation $x = 0.8y$.

  It is INCORRECT to infer that if x is 20% less than y, then y is 20% more than x. Given the statement "x is 80% of y," you should NOT write $y = 1.2x$, but rather $x = 0.8y$, which means $y = \frac{x}{0.8}$ or $y = 1.25x$.

- You can calculate percent increases and decreases using the same equation $A = pB$, where A is the new amount and B is the original amount (read as "the new amount, A, is P percent of the old amount, B," where P is the percentage equivalent of the decimal p). The decimal p that is solved for is relative to 1 (indicating what percent A is of B).

  If $p = 1.2$, there was a 20% growth from B to A (that is, A is 120% of B).

  If $p = 0.85$, there was a 15% decrease from B to A (that is, A is only 85% of B, so B decreased by 15%).

- Percent Change Formula:

$$\text{Percent Change} = \frac{\text{New Value} - \text{Original Value}}{\text{Original Value}}$$

  The decimal is the percent change from the original value, and the sign (positive or negative) tells you whether the change is an increase or decrease.

- If a value changes by a certain percentage and then changes again by a certain percentage, you need to *multiply* the original value by the percentages, NOT add the percentages and then apply the result to the original value, because the reference value changes during the sequence.

  For example, if an item cost $100 initially, but the cost is reduced by 10% (which changes the price to $90), and then you use a 10% off coupon, you are taking 10% off of the new listed price of $90, not the original price. Therefore, the price you pay is 90% of 90% of $100 (you do NOT pay 80% of the original price):

$$\text{price} = 0.9(0.9)(100) = .9(90) = 81$$

Mixtures & Concentrations

- The percentage equations in solution and mixture problems are just linear equations with decimals.

  The amount of a substance in a solution/mixture is equal to the percentage of the solution that is made up of that substance times the total amount of the solution/mixture. For example, if there are A kilograms of a saline solution (a solution of salt in water) that is 10% salt by mass, then $0.1A$ is the amount of salt in that solution.

  When mixing solutions to form a new solution, add the amounts of a substance in the ingredients (the percentage of the solution that is made up of that substance times the amount of the solution) for the two solutions and set them equal to the amount of that substance in the resulting mixture (which, again, is the percentage of the solution that is made up of that substance times the total amount of the solution). For example, if we mix A kilograms of a 10% saline solution and B kilograms of a 20% saline solution to form a new 15% saline solution, we can write the following equation

$$0.1A + 0.2B = 0.15(A + B)$$

  because the amount of salt in the first solution is 10% of the total amount of that solution (which is A kilograms), the amount of salt in the second solution is 20% of the total amount of that solution (which is B kilograms), and the amount of salt in the new solution is 15% of the total amount of that solution (which is $A + B$ kilograms).

Exponential Relationships

Exponential Equations

- Exponential change is characterized by an initial value that is repeatedly multiplied or divided by the same amount.
- Most exponential equations you will have to write or interpret will deal with percent increase (growth) or decrease (decay) and will be of the form

$$y = a \cdot b^x$$

  where a is the initial value, b is the rate of change (the amount that the value is multiplied by over each interval), and x is the number of intervals (usually a time interval).
- In exponential equations where the initial equation's exponent is written in terms of one unit of measurement, but you are supplied with the period or interval variable or value in different units, you should use a proportion showing the relative values of the units to determine the value of the exponent when expressed in the original units.

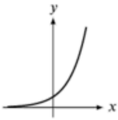
  For example, there are four quarters in one year, so we can use the proportion $\frac{q}{t} = \frac{4}{1}$, where q is the number of quarters and t is the number of years, to convert measurements in one unit to the other.

Graphs of Exponential Equations

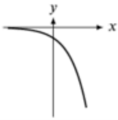
- It is useful to know the general shapes of exponential graphs, where one side of the graph is almost horizontal and the other side is almost vertical. Plugging in test values and graphing a few points should give you a reasonable understanding of the graph for any particular exponential equation.

  In general for equations of the form $y = ab^{cx}$:
  - Positive values of a indicate that the graph has a positive y-intercept and will be entirely above the x-axis.
  - Negative values of a indicate that the graph has a negative y-intercept and will be entirely below the x-axis.
  - When $b \geq 1$ and c is positive AND when $0 < b < 1$ and c is negative, the graph levels off to the left side and goes to infinity (or negative infinity when a is negative) on the right side (the y-value changes slowly, then changes rapidly).

$y = 2(2)^x$ or $y = 2\left(\frac{1}{2}\right)^{-x}$

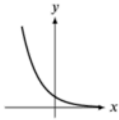


$y = -2(2)^x$ or $y = -2\left(\frac{1}{2}\right)^{-x}$

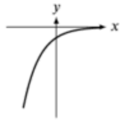


➔ When $b \geq 1$ and c is negative AND when $0 < b < 1$ and c is positive, the graph levels off to the right side and goes to infinity (or negative infinity when a is negative) on the left side (the y-value changes rapidly, then changes slowly).

$y = 2(2)^{-x}$
or
$y = 2\left(\frac{1}{2}\right)^{x}$



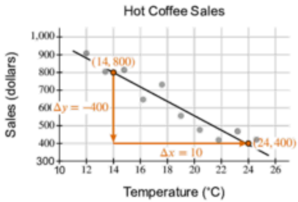
$y = -2(2)^{-x}$
or
$y = -2\left(\frac{1}{2}\right)^{x}$



Scatterplots and Line Graphs

Scatter Plots

- Find the equation of a line of best fit for a scatterplot in the same ways that you did for any regular line. If no line of best fit is drawn, try your best to draw one in that runs roughly through the center of the cluster of points.

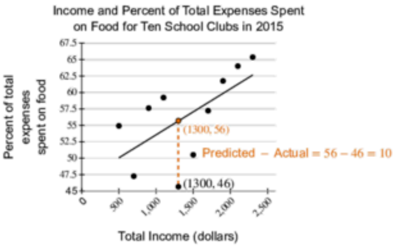


$$m \approx \frac{\Delta y}{\Delta x} = \frac{-400}{10} = -40$$

⇓ Substitute slope and point (14, 800) to solve for b:

$$y = mx + b$$
$$800 = -40(14) + b$$
$$800 - 560 = b$$
$$1{,}360 = b$$
$$\Downarrow$$
$$y = -40x + 1{,}360$$

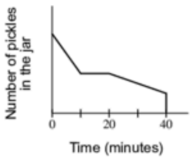
- Find the difference between actual and predicted values by seeing how far above or below the line of best fit a data point is.



- Do not make overly definite statements about what the slopes and intercepts of scatterplots tell us. They are merely models and sometimes only fit very roughly over particular intervals.

Reading Line Graphs

- Most line graph questions are simply related to the slope of the line on certain intervals. Steep lines mean something changed rapidly during an interval. Horizontal lines means something stayed the same for an interval. Vertical lines mean something changed instantaneously.
  For example, Hannah eats pickles while she studies. She eats half of the pickles during the first 10 minutes of studying. After eating half of the pickles, she stops eating for the next 10 minutes. Then she gradually eats the pickles until she purposely spills all of the remaining pickles.



Functions

Evaluating Functions ("Plugging In")

- When plugging into a function $f(x)$, replace each instance of x with the value or expression that is being plugged into the function. Replace each x with a set of blank parentheses as an intermediate step if you have trouble plugging into the function.

$$f(x) = x^2 - 2x + 1 \quad \Rightarrow \quad f(\quad) = (\quad)^2 - 2(\quad) + 1 \quad \Rightarrow \quad \begin{array}{l} f(-3) = (-3)^2 - 2(-3) + 1 \\ f(x+2) = (x+2)^2 - 2(x+2) + 1 \end{array}$$

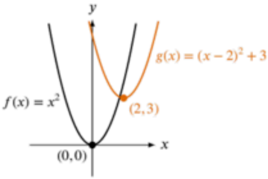
- When working with composite functions, plug the input value into the inside function first and then plug the resulting value into the outside function.

$$\begin{array}{l} g(x) = 4x + 2 \\ h(x) = 3 - g(x) \end{array} \quad \Rightarrow \quad g(0) = 4(0) + 2 = 2 \quad \Rightarrow \quad h(0) = 3 - g(0) = 3 - 2 = 1$$

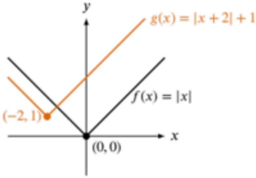
Function Shifting and Unfamiliar Graphs

- The graph of a function $g(x) = f(x - h) + k$ is shifted h units horizontally and k units vertically from the graph of $f(x)$.

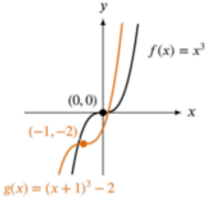
Quadratics



Absolute Value



Cubics



- If you are given an unfamiliar graph type and are asked to find the corresponding function, plug identifiable points, such as the y-intercept or x-intercept, into each of the answer choices, eliminating any that don't fit all of the tested points.

Statistics

Data Sets: Mean, Median, Mode, and Range

- When trying to find the key properties of a data set, you may find it useful to rewrite the data set in order from least to greatest to help prevent mistakes.
- The **mean** is the average of the values in a data set. The mean is found by dividing the sum of the values in the data set by the size of the data set.

$$\text{Mean} = \frac{\text{Sum of Data Set}}{\text{Size of Data Set}}$$

- The **median** is located in the middle of the data set.

  For an odd-sized data set, the position of the median is found by adding 1 to the size of the data set and then dividing by 2. For example, the median of a data set with 49 values is found in position 25 (the 25th value in the data set) because $\frac{49+1}{2} = \frac{50}{2} = 25$.

  For an even-sized data set, the median is found by averaging the two middle values. The positions of these two values are found by dividing the size of the data set by 2; the result gives the position of the first middle value, and the other middle value is found at the next position in the data set. For example, the median of a data set with 100 values is found by averaging the values of the 50th and 51st values in the data set because $\frac{100}{2} = 50$.
- The **mode** of a data set is the value that appears most frequently in a data set.
- The **range** of a data set is the difference between the largest and smallest value in a data set. Subtract the smallest value from the largest value to find the range.

Process	**Set A:** $\{4, 2, 8, 10, 1, 9, 8\}$	**Set B:** $\{2, 0, 0, 4, 5, 5\}$
Increasing Order	$\{1, 2, 4, 8, 8, 9, 10\}$	$\{0, 0, 2, 4, 5, 5\}$
Mean	$\text{Mean}_A = \frac{1+2+4+2(8)+9+10}{7} = \frac{42}{7} = 6$	$\text{Mean}_B = \frac{2(0)+2+4+2(5)}{6} = \frac{16}{6} = \frac{8}{3}$
Median	$A = \{1, 2, 4, 8, 8, 9, 10\}$	$B = \{0, 0, 2, 4, 5, 5\}$ $\text{Median}_B = \frac{2+4}{2} = \frac{6}{2} = 3$
Mode	$\text{Mode}_A = 8$ This value appears twice, and no other value is repeated	No value appears more often than any other: there is **no mode**.
Range	$\text{Range}_A = 10 - 1 = 9$	$\text{Range}_B = 5 - 0 = 5$

Frequency Tables, Histograms, and Standard Deviation

Frequency tables and histograms are used to show how many times particular values occur in a data set. From these representations, we are able to calculate all of the key properties of data sets. To find the mean, we need to find the sum of the data, which can be determined by multiplying each value in the data set by its frequency (or the height of its bar) and summing the results, then dividing by the number of samples. The median is found by counting the number of elements until you reach the position of the median (in the middle of the set).

Ages of 200 People Enrolled in a Hot Yoga Studio

Age	Frequency	
18	34	People 1–34
19	21	People 35–55
23	37	People 56–92
25	38	People 93–130
30	46	
45	24	

$$\text{Mean Age} = \frac{34(18) + 21(19) + 37(23) + 38(25) + 46(30) + 24(45)}{200}$$

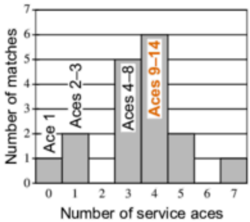
$$\text{Mean Age} = \frac{5{,}272}{200}$$

$$\text{Mean Age} = 26.36$$

There are 200 values in the set, so the median is the average of the 100th and 101st values in the set, both of which are 25 years old.

$$\text{Median Age } = 25$$

Number of Service Aces by a Volleyball Team in 17 Matches



$$\text{Mean Aces} = \frac{1(0) + 2(1) + 5(3) + 6(4) + 2(5) + 1(7)}{17}$$

$$\text{Mean Aces} = \frac{58}{17}$$

$$\text{Mean Aces} \approx 3.41$$

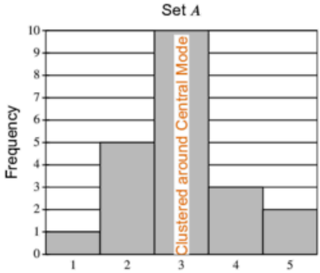
There are 17 values in the set, so the median is the 9th value in the set, which is 4 aces.

$$\text{Median Aces } = 4$$

Standard deviation is a measure of how closely clustered the values in a data set are (how close to the mean of the data most of the values are). Tightly clustered data sets will have a lower standard deviation than will data sets that are more spread out.

Set A	
Value	Frequency
1	1
2	5
3	**10**
4	3
5	2

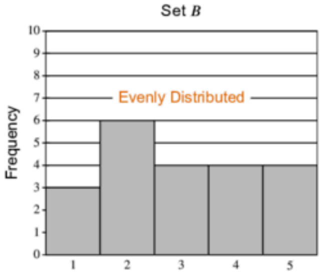
Clustered around Central Mode



Lower Standard Deviation

Set B	
Value	Frequency
1	3
2	6
3	4
4	4
5	4

Evenly Distributed

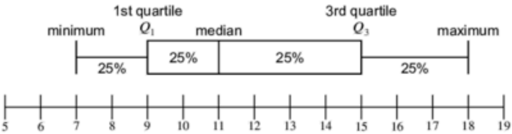


Higher Standard Deviation

Box Plots

- A box plot is formed by drawing short vertical lines for the median and quartiles Q_1 and Q_3 and then connecting the tops and bottoms of those lines to form a box. The outermost vertical lines mark the maximum and minimum values in the set. About 25% of the data set will be found between each vertical line on the box plot (each section contains roughly the same amount of data).

  For the box plot in the figure below, the median is 11; the first quartile is 9, and the third quartile is 15; the minimum is 7, and the maximum is 18.



Survey Design / Interpreting Results

- In order to draw valid conclusions from surveys, the samples must be sufficiently large (so far there has never been a sample that was too small on any released tests because, regardless of the population size, even seemingly small sample sizes are fairly accurate), and **most importantly, truly random**. If the subjects in a survey share some trait (other than just being members of the larger population), then the survey's results are restricted to just the population that shares that trait and cannot be applied to the larger population to which those subjects belong.
- The **margin of error** accounts for the potential difference between the true value for an entire population and the value found based on a survey sample. The true value for the whole population is most likely (but not definitely) found inside the margin of error around the value found for the survey. A larger sample size leads to a smaller margin of error.

  An ecologist selected a random sample of 50 beavers from a river and found that the mean weight of the beavers in the sample was 42 pounds (lbs), with an associated margin of error of 3.1 lbs. Therefore, any weight between $42 - 3.1 = 38.9$ lbs and $42 + 3.1 = 45.1$ lbs is a plausible value for the mean weight of all the beavers in the river.
- Do not make overly definitive claims based on a survey. The results only apply to the population that shares the traits common to those sampled. A cause-and-effect relationship is highly unlikely unless you are specifically told that all other variables have been controlled for. You are a lot safer merely making a claim just that there is a correlation between two things (rather than a cause-and-effect relationship).

