Lines

Slope-Intercept Form Linear Equations

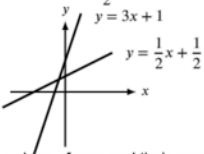
- Slope-Intercept Form Linear Equation:

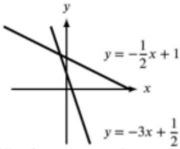
$$y = mx + b$$

- Total Amount = (Amount per Thing × Number of Things) + Starting Amount
  Example: Jabrill has \$5 (starting amount) and earns \$4 per shirt sold $\Rightarrow$ $j = 4s + 5$
- The four components of a linear equation in the form $y = mx + b$ are the y-value, the x-value, the coefficient m (slope), and the constant b (y-intercept). We can solve for any one of these when we know the other three.
- Slope Formula:

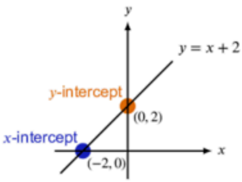
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

- A positive slope means that the y-value increases as the x-value increases. Steeper lines have higher slopes because the y-value goes up more as the x-value increases. A line with a slope of 3 is steeper than a line with a slope of $\frac{1}{2}$.
  A negative slope means that the y-value decreases as the x-value increases. Steeper lines have lower (more negative) slopes because the y-value decreases more as the x-value increases. A line with a slope of -3 is steeper than a line with a slope of $-\frac{1}{2}$.





- Horizontal lines have a slope of zero and their equations are of the form $y = c$, where c is a constant (the y-value is the same for all points on the line).
  Vertical lines have an undefined slope and their equations are of the form $x = c$, where c is constant (the x-value is the same for all points on the line).
- The y-intercept is the starting amount or the value of y when the x-value is 0.
  The x-intercept is the value of x when the y-value is equal to 0.



- You can find the equation of any line as long as you know any two points on the line or know the slope and any one point. If you are given the slope and y-intercept, then you already have everything you need to make the equation.

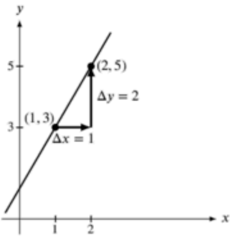
  Step 1: Use the Slope Formula to find the slope from two points: $(1,3)$ and $(2,5)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{2-1} = \frac{2}{1} = 2$$

  Step 2: Plug the slope and a point into the Slope-Intercept Form to find the y-intercept: $(1,3)$ and $m = 2$

$$y = mx + b \quad \Rightarrow \quad 3 = 2(1) + b \quad \Rightarrow \quad 3 = 2 + b \quad \Rightarrow \quad 1 = b \quad \Rightarrow \quad \boxed{y = 2x + 1}$$

- Using up-and-over arrows with graphical representations to find Δy and Δx, or counting differences between coordinate values between rows of tables, is often less error prone than substituting the coordinates of points into the long form of the Slope Formula.



$\Delta x = 1$ $\quad$ $\Delta y = 2$

x	y
1	3
2	5
3	7
4	9

- When asked to interpret the meaning of a coefficient or constant in a Slope-Intercept Form linear equation, first determine which one of the four components of a linear equation it is (the y-value, the slope, the x-value, or the y-intercept).

  The y-intercept is the "starting amount" and is measured in the same units as the y-value.

  The slope is the rate of change of the y-value as the x-value changes and is measured in the units of the y-value divided by the units of the x-value (look for keywords like "per" which indicate division).

Standard Form Linear Equations

- Standard Form Linear Equation:

$$Ax + By = C$$

- Convert Standard Form to Slope-Intercept Form:

$$y = \frac{-A}{B}x + \frac{C}{B}$$

- Slope of a Line in Standard Form:

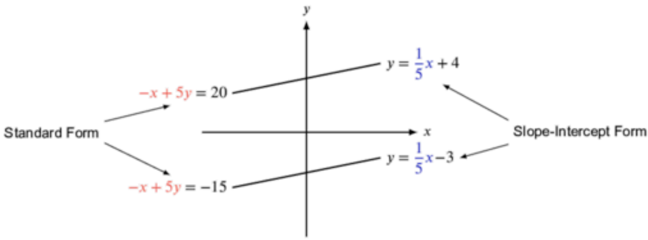
$$\frac{-A}{B}$$

- y-intercept of a Line in Standard Form:

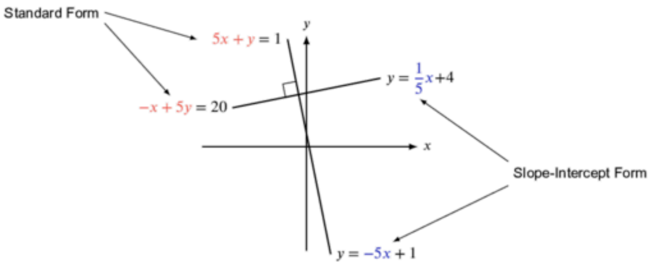
$$\frac{C}{B}$$

Parallel and Perpendicular Lines

- Parallel lines are the same distance apart everywhere and thus they never intersect. Parallel Lines have the same slope but different y-intercepts.
- Parallel lines in Standard Form will have the same x- and y-coefficients (or the ratio of x- and y-coefficients will be the same).



- Perpendicular lines cross at a right angle. The slope of each line in a pair of perpendicular lines is the **negative reciprocal** of the slope of the other line.
- Perpendicular lines in Standard Form will have the x- and y-coefficients swapped, and one of the coefficients will be multiplied by -1 (or the ratio of the x- and y-coefficients will be flipped and multiplied by -1).



Systems of Linear Equations

Substitution

- Solutions to systems of linear equations can be thought of as points where the graphs of the lines intersect. At these intersection points, both lines have the same x- and y- values.

- Set expressions for y equal to each other to find the solution point when both equations are already in Slope-Intercept Form. Use substitution when the equations are in different forms, particularly when one variable is already solved for in terms of the other.

$$\left.\begin{array}{l} y = 2x+1 \\ y = x+4 \end{array}\right\} \Rightarrow \quad 2x+1 = -x+4 \quad \Rightarrow \quad 3x = 3 \quad \Rightarrow \quad x = 1$$

$$\left.\begin{array}{r} x+4y = 20 \\ x = 4y \end{array}\right\} \Rightarrow \quad (4y)+4y = 56 \quad \Rightarrow \quad 8y = 56 \quad \Rightarrow \quad y = 7$$

Elimination

- Use elimination to solve most systems of linear equations, particularly when both equations are in Standard Form. Multiply one or both equations by numbers that will cause the coefficients of the variable you want to eliminate to cancel out when the equations are combined through addition or subtraction.

Solve for f:

$$\left.\begin{array}{r} f+s = 30 \\ 4f+6s = 140 \end{array}\right\} \Rightarrow \left.\begin{array}{r} -6(f+s) = -6(30) \\ 4f+6s = 140 \end{array}\right\} \Rightarrow \begin{array}{r} -6f-6s = -180 \\ +\quad 4f+6s = 140 \\ \hline -2f \qquad = -40 \end{array} \Rightarrow \quad f = 20$$

Solve for x:

$$\left.\begin{array}{r} -5x+2y = 10 \\ 4x+5y = 25 \end{array}\right\} \Rightarrow \left.\begin{array}{r} -5(-5x+2y) = -5(10) \\ 2(4x+5y) = 2(25) \end{array}\right\} \Rightarrow \begin{array}{r} 25x-10y = -50 \\ +\quad 8x+10y = 50 \\ \hline 33x \qquad = 0 \end{array} \Rightarrow \quad x = 0$$

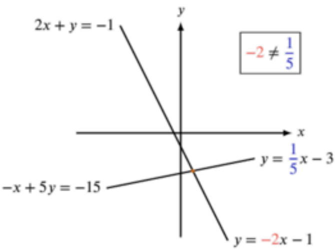
- Use combination without eliminating either variable if possible when the problem asks for an expression involving terms with both variables.

Find $x+y$:

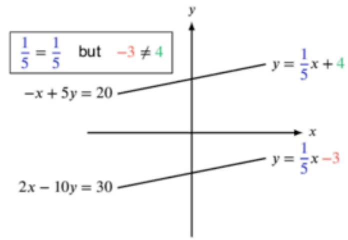
$$\begin{array}{r} 2x+3y = 100 \\ +\quad 4x+3y = 380 \\ \hline 6x+6y = 480 \end{array} \Rightarrow \quad x+y = 80$$

Number of Solutions to Systems of Linear Equations

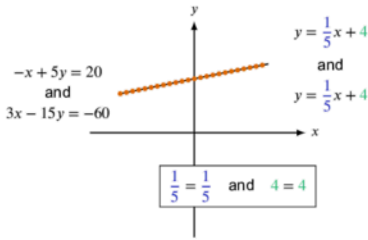
- A system of linear equations has **one solution when the two lines are *not* parallel (have different slopes) and thus cross at one point**.



➔ A system of linear equations has **no solutions when the lines are parallel** (have the same slopes and different y-intercepts), because the lines do not intersect.



➔ A system of linear equations has **infinite solutions when the two lines are exactly the same**. The **slopes and y-intercepts must be the same**.



➔ When asked to find unknown coefficient values in systems of linear equations in Standard Form that have no solutions or infinitely many solutions, use proportions of the ratios of the coefficients and constants to solve for unknown values.

$$\text{No Solutions}: \quad \begin{array}{c} 12x - 4y = 2 \\ ax + 2y = 7 \end{array} \;\Rightarrow\; \frac{a}{12} = \frac{2}{-4} \;\Rightarrow\; a = 12\left(\frac{-1}{2}\right) \;\Rightarrow\; a = -6$$

$$\infty\ \text{Solutions}: \quad \begin{array}{c} 4x + 6y = 12 \\ -2x - 3y = b \end{array} \;\Rightarrow\; \frac{b}{12} = \frac{-3}{6} \;\Rightarrow\; b = 12\left(\frac{-1}{2}\right) \;\Rightarrow\; b = -6$$

Alternatively, multiply one or both equations by scale factors so that any known coefficients or constants in the same position can be matched between the two equations.

$$\text{No Solutions}: \quad \begin{array}{c} 8x - 2y = 1 \\ ax + 3y = 3 \end{array} \;\Rightarrow\; \begin{array}{c} -3(8x - 2y) = -3(1) \\ 2(ax + 3y) = 2(3) \end{array} \;\Rightarrow\; \begin{array}{c} -24x + 6y = -3 \\ 2ax + 6y = 6 \end{array} \;\Rightarrow\; \begin{array}{c} 2a = -24 \\ a = -12 \end{array}$$

$$\infty\ \text{Solutions}: \quad \begin{array}{c} 3x - 9y = 12 \\ ax + by = 4 \end{array} \;\Rightarrow\; \begin{array}{c} 3x - 9y = 12 \\ 3(ax + by) = 3(4) \end{array} \;\Rightarrow\; \begin{array}{c} 3x - 9y = 12 \\ 3ax + 3by = 12 \end{array} \;\Rightarrow\; \begin{array}{c} 3a = 3 \\ a = 1 \end{array} \text{ and } \begin{array}{c} 3b = -9 \\ b = -3 \end{array}$$

Linear Inequalities & Absolute Value

Linear Inequalities

- The pointy or small end of the inequality sign is the lesser side of the inequality; the open or big end of the inequality sign is the greater side of the inequality.

Name	Symbol	Usage
Less Than	$<$	The value on the left is less than the value on the right. The inequality $x < 5$ means that x can be any value from $-\infty$ up to, but **not** including, 5.
Less Than Or Equal To	$\leq$	The value on the left is less than or equal to (no more than) the value on the right. For example, $x \leq 5$, means that x can be any value from $-\infty$ up to 5, including 5 itself.
Greater Than	$>$	The value on the left is greater than the value on the right . The inequality $x > 5$ means that x can be any value greater than 5 up to ∞.
Greater Than Or Equal To	$\geq$	The value on the left is greater than or equal to (no less than) the value on the right. For example, $x \geq 5$ means that x can be any value from 5 (including 5 itself) up to ∞.
Absolute Value	$\lvert a \rvert$	Produces a non-negative value indicating how far a number is from 0. ($\lvert -5 \rvert = 5$)

- Solve inequalities like you solve equations, but remember to flip the sign if you multiply or divide by a negative number.

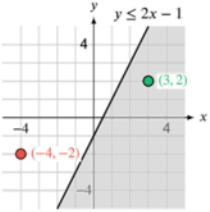
$$-8x + 3 \leq 59 \quad \Rightarrow \quad -8x \leq 56 \quad \Rightarrow \quad x \geq -7$$

- In real world problems, correct answers often have to be integers, particularly if the problem involves items that cannot be divided into fractional amounts (books, shirts, etc.).
- Compound inequalities can be solved in one step as long as you are careful about flipping both inequality signs if you multiply or divide by negative numbers. Just make sure to apply the same operation to all three parts of the compound inequality.

$$2 < -3x + 2 < 8 \quad \Rightarrow \quad 0 < -3x < 6 \quad \Rightarrow \quad \frac{0}{-3} > x > \frac{6}{-3} \quad \Rightarrow \quad 0 > x > -2$$

- When graphing linear inequalities in the xy-plane, use a solid line if there is a $\leq$ or $\geq$ sign in the inequality (because points on the line **are** solutions to the inequality), and use a dashed line if there is a $<$ or $>$ sign in the inequality (because points on the line **are NOT** solutions to the inequality). Shade **above** the line to show the solution region when the y-variable is **greater than** the linear expression. Shade **below** the line to show the solution region when the y-variable is **less than** the linear expression.

  Plug in the x- and y-coordinates of a point to the inequality to see if a point falls in the shaded solution region or is on a solid boundary line.

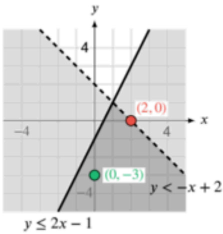


$y \leq 2x - 1$

Test $(3, 2)$	Test $(-4, -2)$
$2 \leq 2(3) - 1$	$-2 \leq 2(-4) - 1$
$2 \leq 6 - 1$	$-2 \leq -8 - 1$
✓ $2 < 5$	✗ $-2 \nleq -9$

Systems of Linear Inequalities

- Solutions to a system of linear inequalities must satisfy all the inequalities in the system. If the inequalities are graphed on the same xy-plane, the solutions are located within the overlapping shaded regions and on any solid lines bounding those regions.



$$y \leq 2x - 1$$
$$y < -x + 2$$

Test $(0, -3)$	Test $(2, 0)$
$-3 \leq 2(0) - 1$	$0 \leq 2(2) - 1$
$-3 < -(0) + 2$	$0 < -(2) + 2$
$\Downarrow$	$\Downarrow$
✓ $\mathbf{-3 \leq -1}$	$0 \leq 4 - 1$
✓ $\mathbf{-3 < 2}$	$0 < -2 + 2$
	$\Downarrow$
	✓ $\mathbf{0 \leq 3}$
	✗ $\mathbf{0 \nless 0}$

Absolute Value

- You can solve absolute value equations by making two equations: one where the expression in the absolute value bars is set equal to the original value on the other side of the equation, and one where the expression is set equal to the negative of that value. For example, $|x| = 5$ yields both $x = 5$ and $x = -5$.

  Note that while the expression *within* the absolute value bars can have any value, positive or negative, **the absolute value of the expression can never be negative, by definition**. For example, while $|x| = 5$ yields two possible solutions ($x = 5$ and $x = -5$), the equation $|x| = -5$ yields **zero solutions** because the absolute value of an expression can never be negative.

- $|x + a| = b \quad \Rightarrow \quad \begin{cases} x + a & = b \\ x + a & = -b \end{cases}$

- $|x + a| > b \quad \Rightarrow \quad \begin{cases} x + a & > b \\ x + a & < -b \end{cases}$

- $|x + a| < b \quad \Rightarrow \quad \begin{cases} x + a & < b \\ x + a & > -b \end{cases} \Rightarrow \quad -b < x + a < b$

Exponents

Exponent Rules

- Exponential terms consist of a base a and exponent b and are in the form a^b.
- Exponents tell you how many factors of the base should be multiplied by each other. For example, $2^5 = 2 \times 2 \times 2 \times 2 \times 2$ (there are 5 factors of 2 multiplied by each other).
- $a^b a^c = a^{b+c}$
- $\left(a^b\right)^c = a^{bc}$
- $\dfrac{a^b}{a^c} = a^{b-c}$
- $a^{-b} = \dfrac{1}{a^b}$ and $\dfrac{1}{a^{-b}} = a^b$
- $(ab)^c = a^c b^c$ BUT WATCH OUT FOR THIS MISTAKE: $(a+b)^c \neq a^c + b^c$
- $a^{\frac{b}{c}} = \sqrt[c]{a^b} = \left(\sqrt[c]{a}\right)^b$ Note that the rule will hold as long as the base, a, is non-negative. Rewrite radical expressions as terms with fractional exponents.

Methods of Solving Exponent Equations

- You can solve exponential equations by raising both sides to a power such that the exponents of the variable term will multiply to the correct power (the power you are looking for) on one side of the equation. For example if you are given the equation $x^{\frac{1}{5}} = 2$, you can solve for x^1 by raising both sides of the equation to the fifth power:

$$\left(x^{\frac{1}{5}}\right)^5 = 2^5 \quad \Rightarrow \quad x = 32$$

- You can write relatively large bases in terms of smaller bases in order to rewrite exponential terms with fractional exponents (without using a calculator) in order to match answer choices.

  For example, if we wanted to simplify the term $9^{\frac{3}{4}}$, we could rewrite the 9 as 3^2, allowing us to break the exponent up and rewrite $9^{\frac{3}{4}}$ as a radical expression with a base of 3.

$$9^{\frac{3}{4}} = \left(3^2\right)^{\frac{3}{4}} = 3^{2\left(\frac{3}{4}\right)} = 3^{\frac{3}{2}} = 3^{\left(1+\frac{1}{2}\right)} = 3 \cdot 3^{\frac{1}{2}} = 3\sqrt{3}$$

  It is also advantageous to write terms in another base whenever you have two different base numbers that share a common factor.

  For example, if we are given the equation $27^x \cdot 3^{4y} = 3^5$ and asked to solve for $3x + 4y$, we can combine the terms in the expression on the left side if we rewrite 27 as a base of 3 raised to the third power, allowing us to apply exponent rules to combine the terms.

$$27^x \cdot 3^{4y} = 3^5 \quad \Rightarrow \quad \left(3^3\right)^x \cdot 3^{4y} = 3^5 \quad \Rightarrow \quad 3^{3x} \cdot 3^{4y} = 3^5 \quad \Rightarrow \quad 3^{3x+4y} = 3^5$$

  Changing the base was useful in this example because it allowed us to write both sides of the equation as 3 raised to a power, which allows us to very easily see that $3x + 4y$ must be equal to 5 (since 3 to one power can't be equal to 3 to a different power).

Quadratics & Other Polynomials

Standard Form Polynomials

- The Standard Form for a univariate (single variable) polynomial puts the terms in order from highest to lowest power, so the Standard Form of a quadratic (second-degree polynomial) equation is

$$y = ax^2 + bx + c$$

where a, b, and c are constants.

- When adding or subtracting expressions, group and combine like terms of the same variable (same base and exponent).

$$(3x + 1 - 2y^2) - (2x + 4 - 5y^2)$$
$$3x - 2x + 1 - 4 - 2y^2 + 5y^2$$
$$x - 3 + 3y^2$$

- When multiplying expressions, distribute all terms in the first expression to all terms in the second expression.

$$(2x + 3)(x + 2) = 2x(x + 2) + 3(x + 2)$$
$$(2x + 3)(x + 2) = 2x(x) + 2x(2) + 3(x) + 3(2)$$
$$(2x + 3)(x + 2) = 2x^2 + 4x + 3x + 6$$
$$(2x + 3)(x + 2) = 2x^2 + 7x + 6$$

Solving Polynomial Equations

- If the two sides of an equation are polynomials written in Standard Form, and you are told that the equation is true for all values of the variable (that is, the polynomials are **equivalent**), you can match the coefficients of corresponding terms to find the values of any unknown coefficients.

$$-2x(5x + 1) + 6(5x + 1) = ax^2 + bx + c$$
$$-10x^2 - 2x + 30x + 6 = ax^2 + bx + c$$
$$-10x^2 + 28x + 6 = ax^2 + bx + c \Rightarrow$$

$a = -10$, $b = 28$, and $c = 6$

Factoring

- When factoring a quadratic expression in Standard Form where the coefficient of the x^2 term is 1, $x^2 + bx + c = 0$, we need to find two numbers, p and q, such that $p + q = b$ (the sum of the numbers is b) and $pq = c$ (the product of the numbers is c). The solutions to the equation are $-p$ and $-q$, and we can rewrite the expression in Factored Form. One of the factors is $(x + p)$ and the other factor is $(x + q)$.

$$(x + p)(x + q) = 0$$

For example, to find the factors of the quadratic expression in the equation $y = x^2 + 3x + 2$, we need to find two numbers that add to 3 and multiply to 2. The numbers 1 and 2 add to 3 (we know that $1 + 2 = 3$) and multiply to 2 (we know that $1(2) = 2$), so the factors are $(x + 1)$ and $(x + 2)$. The solutions are thus -1 and -2.

$$y = x^2 + 3x + 2$$
$$y = (x + 1)(x + 2)$$

- The sum of the solutions of a quadratic equation in Standard Form, $ax^2 + bx + c = 0$, is equal to

$$\frac{-b}{a}$$

- The product of the solutions of a quadratic equation in Standard Form, $ax^2 + bx + c = 0$, is equal to

$$\frac{c}{a}$$

Other Methods of Finding the Roots of Quadratics

- When the coefficient of the x^2 term is NOT 1 (when $a \neq 1$), first try dividing all of the terms by a to see if the resulting expression is easily factorable.

$$3x^2 - 12x - 15 = 0$$

$$\frac{3x^2 - 12x - 15}{3} = \frac{0}{3}$$

$$x^2 - 4x - 5 = 0$$

$$(x-5)(x+1) = 0$$

$$x = 5\,,\, x = -1$$

- Sometimes, when the coefficient of the x^2 term is NOT 1 (when $a \neq 1$), you can use the Product Sum (ac) Method, which is a generalized form of factoring.
When factoring a quadratic expression in Standard Form $ax^2 + bx + c = 0$ using the Product Sum (ac) Method, we need to find two numbers, r and s, such that $r + s = b$ (the sum of the numbers is b) and $rs = ac$ (the product of the numbers is ac). The solutions to the equation are $\frac{-r}{a}$ and $\frac{-s}{a}$, and we can rewrite the expression in Factored Form. One of the factors is $\left(x + \frac{r}{a}\right)$ and the other factor is $\left(x + \frac{s}{a}\right)$.

$$a\left(x + \frac{r}{a}\right)\left(x + \frac{s}{a}\right) = 0$$

$$4x^2 - x - 3 = 0$$

Product: $ac = 4(-3) = -12$
Sum: $b = -1$
The numbers r and s whose sum is -1 and whose product is -12 are 3 and -4, so $r = 3$ and $s = -4$.
The zeros are $x = \frac{-r}{a} = \frac{-3}{4}$ and $x = \frac{-s}{a} = \frac{-(-4)}{4} = 1$.
The factors are $\left(x + \frac{3}{4}\right)$ and $(x - 1)$.

- If you cannot find numbers that satisfy the conditions for factoring, you can always use Completing the Square:

1. Divide both sides of the equation by the a coefficient to eliminate the coefficient of the x^2 term.
2. Move the constant to other side to isolate the x terms.
3. Replace the left side of the equation with $\left(x + \frac{b}{2}\right)^2$ (where b is the coefficient of the x term) and balance the extra constant on the other side by adding $\left(\frac{b}{2}\right)^2$ to the right side of the equation, pre-calculating $\frac{b}{2}$ so you don't have to do that twice.
4. Take the square root of both sides, being careful to include both positive and negative square roots.
5. Move the constant to right side of the equation to isolate and solve for x.
6. Enumerate the solutions.