Unit Conversions

Simple Conversions with Proportions

- When converting from one unit to another unit based on a direct conversion rate, you can simply use a proportion to solve the problem.

  EXAMPLE:
  Convert 3 inches to centimeters ($1 \text{ inch } = 2.54 \text{ cm}$):

$$\frac{\ell_{\text{cm}}}{\ell_{\text{in}}} = \frac{2.54\text{ cm}}{1\text{ in}} \quad \Rightarrow \quad \frac{\ell_{\text{cm}}}{3\text{ in}} = \frac{2.54\text{ cm}}{1\text{ in}} \quad \Rightarrow \quad \ell_{\text{cm}} = \left(\frac{2.54\text{ cm}}{1\cancel{\text{ in}}}\right)(3\cancel{\text{ in}}) \quad \Rightarrow \quad \ell_{\text{cm}} = 7.62\text{ cm}$$

Factor-Label Method

- Use the Factor-Label method when a direct conversion from one unit to another is not available. String together unit conversion factors in order to cancel one unit at a time until the desired units are achieved.

  EXAMPLE:
  Convert 76 inches to meters ($1 \text{ inch } = 2.54 \text{ cm}$ and $1 \text{ m} = 100 \text{ cm}$):

$$76\cancel{\text{ in}}\left(\frac{2.54\cancel{\text{ cm}}}{1\cancel{\text{ in}}}\right)\left(\frac{1\text{ m}}{100\cancel{\text{ cm}}}\right) = 1.93\text{ m}$$

- Identify a **starting point** for word problems with unit conversions by picking out information that tells us about the **end goals**. In order to determine the steps needed after the starting point is identified, try to cancel or convert unwanted units, one-by-one, based on additional information in the problem.

The Distance-Speed-Time Equation

- The distance-speed-time equation is

$$d = st$$

  where d is distance, s is speed, and t is time.

Angles, Triangles, and Trigonometry

Angles

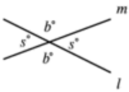
- Angles are formed by the intersection of two lines; the point at which the two lines meet is called the **vertex** (plural, **vertices**).
- Angles are most commonly measured in **degrees**, which is denoted by the degree symbol (°).
- A 90°angle is a **right angle** (the lines forming the angle are perpendicular). Right angles are often marked with a small square instead of an arc and angle measurement.

Any angles that together form a right angle sum to 90°. When two angles form a 90° angle they are called **complementary angles**.

- A 180° angle results in a straight line.

Any angles that together form a straight line sum to 180°. When two angles form a 180° angle they are called **supplementary angles**.

- Angles that measure more than 0° and less than 90° are called **acute angles**. Angles that measure more than 90° and less than 180° are called **obtuse angles**. Angles greater than 180° are called **reflex** angles.
- Whenever two lines intersect, four angles are formed, and the angles opposite each other, called **Vertical Angles**, are equal.

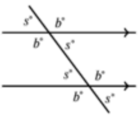


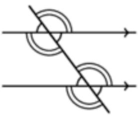
As previously explained, angles that form a line sum to $180°$. Therefore, $s + b = 180$.

- To indicate congruence (equality of measure) for sides and angles, those features can be marked with dashes or arcs, called hatch (or hash or tick) marks. Sides marked with the same number of hatch marks are congruent, and angles marked with the same number of arcs are congruent.

  Parallel lines can be marked by arrow heads. Lines marked with the same number of arrow heads are parallel.

- When a diagonal (non-perpendicular) line runs across two parallel lines, eight angles are formed. The four larger (obtuse) angles are all equal to each other. Similarly, the four smaller (acute) angles are also equal to each other.

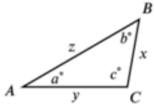




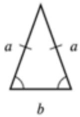
Triangles and Other Polygons

- **The interior angles of any triangle sum to 180°**. The length of each side is positively correlated with the size of the angle opposite that side, so the **largest side of a triangle is always across from the largest angle of the triangle, and the smallest side is across from the smallest angle**.

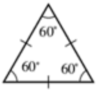
  In the triangle below, $a + b + c = 180$ because the sum of any triangle's interior angles is 180°. Since $\angle C$ is the largest angle, z is the longest side. Similarly, since $\angle A$ is the smallest angle, x is the shortest side.



- **In isosceles triangles, two of the angles are equal to each other, and the sides across from those angles are therefore also equal to each other.**



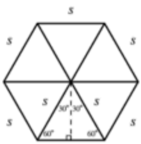
- **In equilateral triangles, all of the sides are equal to each other, and therefore all of the angles are also equal**. Because all three of the angles are equal and they must sum to 180°, each angle is $\frac{180^\circ}{3} = 60^\circ$.



- For a polygon with n sides and angles, the sum of the interior angles can be found using the following formula, which is very rarely needed:

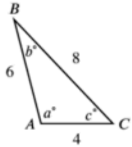
$$\text{Sum of Interior Angles} = 180(n - 2)$$

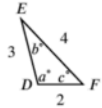
- In a **regular polygon**, all sides are equal, and all angles are equal. The measure of any angle is equal to the sum of the interior angles divided by the number of angles.
- **Regular hexagons** (six-sided polygons) can be split into 6 equilateral triangles if we draw a line from the center of the figure to each of the six vertices.



Similar Triangles

- Two triangles are called **Similar Triangles** when they have the same interior angles. Put another way, each angle in one triangle has a corresponding match in the other triangle.





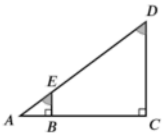
In the figure above, $\triangle ABC$ is similar to $\triangle DEF$ because all of the angles in $\triangle ABC$ have a matching angle in $\triangle DEF$. Notice that the triangles are different sizes, however. **In similar triangles, all of the angles have a match, but the sides are not *necessarily* equal. However, sides across from matching angles are always in the same proportion to each other.**

In the figure shown above, $\overline{AB}$ corresponds to $\overline{DE}$; $\overline{BC}$ corresponds to $\overline{EF}$; and $\overline{AC}$ corresponds to $\overline{DF}$. In these particular triangles, the sides in $\triangle ABC$ are all twice the length of the corresponding sides in $\triangle DEF$.

- **If we know (or can determine) that there are two angles that are the same in two triangles, the third angle must also be the same** because the three angles in each triangle must sum to the same value (180°). This, in turn, means that **the two triangles are similar triangles**.

- Similar triangles are commonly depicted as one triangle inside of another, where you can show that the triangles must be similar because all the angles in both triangles will have a match.

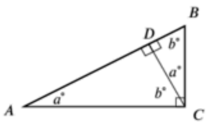
In the figure below, $\triangle ACD$ is similar to $\triangle ABE$ because they both contain $\angle A$ and a right angle, which means the third angles in the two triangles must also be equal.



- You can use part-to-whole or part-to-part ratios to solve for lengths in divided triangles. When a **triangle is divided by a line parallel to one side, the other two sides are divided proportionally**; this is the **Triangle Proportionality Theorem**.

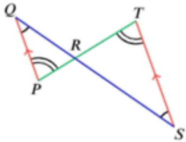
In the triangle above, the Triangle Proportionality Theorem tells us that $\frac{AE}{ED} = \frac{AB}{BC}$.

- Right triangles can be divided with a single line that creates three similar triangles. In the figure below, a line is drawn from the right angle vertex C that intersects the hypotenuse at a right angle at point D. The larger right triangle is split into two smaller right triangles, ΔADC and ΔCDB, both of which are similar to the original triangle ΔACB and thus to each other as well.



- Similar triangles arise in arrangements such as that shown in the figure below, with the two similar triangles touching at one vertex (where there is a vertical angle) and each having a side that is parallel to a side in the other triangle.

  In the figure below, the angles meeting at point R are vertical angles (and thus are equal to each other), and side $\overline{PQ}$ is parallel to side $\overline{ST}$.



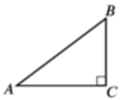
$$\Delta PQR \sim \Delta TSR$$
and
$$PQ : QR : RP = TS : SR : RT$$

Trigonometry

- In a right triangle, the side that is across from the right angle is called the **hypotenuse** ($\overline{AB}$ in the triangle below).

  The side across from an acute angle is called the **opposite** side. In the triangle below, side $\overline{BC}$ is the opposite side for $\angle A$, and side $\overline{AC}$ is the opposite side for $\angle B$.

  The side that forms an angle with the hypotenuse is called the **adjacent** side of that angle. In the triangle below, side $\overline{AC}$ is the adjacent side for $\angle A$, and side $\overline{BC}$ is the adjacent side for $\angle B$.



$$\sin A = \frac{BC}{AB} \quad \sin B = \frac{AC}{AB}$$
$$\cos A = \frac{AC}{AB} \quad \cos B = \frac{BC}{AB}$$
$$\tan A = \frac{BC}{AC} \quad \tan B = \frac{AC}{BC}$$

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

- The trig ratios can be remembered using the mnemonic **SOH-CAH-TOA**, which distills the information that **S**ine is **O**pposite over **H**ypotneuse, **C**osine is **A**djacent over **H**ypotenuse, and **T**angent is **O**pposite over **A**djacent.
- Trigonometric functions' values are derived from the side ratios **in a right triangle**, and their **values for a given angle don't change** if the angle appears in a figure that is not a right triangle; the **sine of an angle is the sine of that angle regardless of what type of figure** includes that angle. However, you can **only calculate these functions from a figure when the angle is part of a right triangle**.
- The sine and cosine of complementary angles have the following relationship:

$$\sin(x^\circ) = \cos(90^\circ - x^\circ) \quad \text{and} \quad \cos(x^\circ) = \sin(90^\circ - x^\circ)$$

  This is useful because the acute angles in a right triangle sum to 90°.

Perimeter & Area

- The **perimeter** of a polygon is the sum of the lengths of the outside edges.
- **Area** is a measure of the space that a shape covers on a plane. Area is measured in units of length squared.
- The area of a rectangle of length ℓ and width w is given by

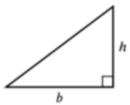
$$A = \ell w$$

The area of a square of side length s is given by

$$A = s^2$$

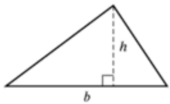
- For a triangle with a base length b and height h, the area A is found with the following formula:

$$A = \frac{1}{2}bh$$

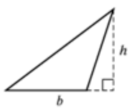


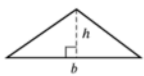
- The height of a triangle is always perpendicular to the base.

To show the height for acute triangles (all angles smaller than 90°), you usually have to draw in a line that is perpendicular to the base and that divides the triangle into two right triangles.



For obtuse triangles (one angle is larger than 90°), depending on the orientation, you may need to draw the perpendicular line for the height outside of the triangle as in the first picture below. Alternatively, you can always rotate the triangle (as in the second picture below) so that the perpendicular height line goes into the largest angle.





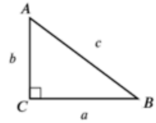
- If you need to find the area of other polygons, divide them into triangles and rectangles so that you can sum the areas of the individual sections.

Pythagorean Theorem & Special Right Triangles

- If you know two of the side lengths of a right triangle, you can use the **Pythagorean Theorem** to find the length of the third side.

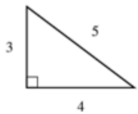
  The Pythagorean Theorem states the following for a right triangle with legs measuring a units and b units and hypotenuse measuring c units:

$$a^2 + b^2 = c^2$$



- There are some special right triangles in which all three sides of the triangle are whole numbers (or are in whole number ratios with each other). These are called **Pythagorean Triples**.
- The most common Pythagorean Triple is the $\mathbf{3:4:5}$ **triangle**.

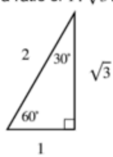
  In a right triangle with legs of length 3 and 4, the hypotenuse has length 5.



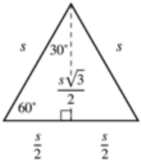
$$\begin{aligned} a^2 + b^2 &= c^2 \\ 3^2 + 4^2 &= 5^2 \\ 9 + 16 &= 25 \\ 25 &= 25 \end{aligned}$$

  The actual side lengths can be any set of values that conform to the $3:4:5$ ratio.

- Other Pythagorean Triples include $\mathbf{5:12:13}$ triangles (occasionally seen), and the far rarer $\mathbf{7:24:25}$ triangles, $\mathbf{8:15:17}$ triangles, and $\mathbf{20:21:29}$ triangles.
- The test makes extensive use of **angle-based special right triangles** because their sides are in easy-to-write-and-remember ratios to each other.
- The first important angle-based special right triangle is a **30-60-90 right triangle**, which has a 30°, 60°, and 90° angle. Its sides, from shortest to longest, are in a ratio of $1:\sqrt{3}:2$.

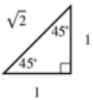


One of the most common ways that 30-60-90 triangles are hidden is in equilateral triangles. If you divide an equilateral triangle with side lengths s in half by drawing a line from any vertex to its opposite side and intersecting that opposite side at a right angle, the triangle is divided into two 30-60-90 triangles where the short legs of these triangles have length $\frac{s}{2}$ and the height of the triangles is $\frac{s\sqrt{3}}{2}$.

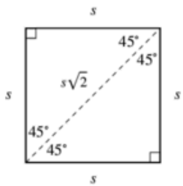


Note that this can be useful when dealing with regular hexagons, which can be divided into 6 equilateral triangles, which can then be divided into 30-60-90 triangles.

The second important angle-based special right triangle is a **45-45-90 right triangle**. Since **two angles are the same, these triangles are isosceles, meaning that both legs are the same length**. The sides are in a ratio of $1 : 1 : \sqrt{2}$.



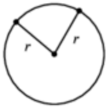
One of the most common ways that 45-45-90 triangles are hidden is in squares. If you divide a square with side lengths s in half diagonally from one vertex to its opposite, the square is divided into two 45-45-90 triangles where the legs have length s and the hypotenuse has length $s\sqrt{2}$.



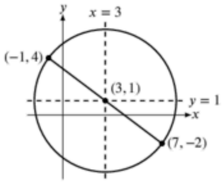
Circles & Volume

Circles & Angles

- A circle is a shape formed by **all of the points that are the same distance away from one central point** (the center of the circle). The **radius**, marked r in the figure below, is the distance from any point on the circle to the center point.



- A **diameter** is a line from one point on the arc to its opposite point on the arc that goes directly through the center of the circle and whose length, d, is equal to twice the radius: $d = 2r$.
- The center point of the circle is the midpoint between the endpoints of any diameter. This center point's x- and y-coordinates can be found by averaging the x- and y-coordinates, respectively, of the endpoints of any diameter.



$$x_{center} = \frac{(-1)+7}{2} \qquad y_{center} = \frac{4+(-2)}{2}$$

$$x_{center} = \frac{6}{2} \qquad y_{center} = \frac{2}{2}$$

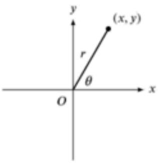
$$x_{center} = 3 \qquad y_{center} = 1$$

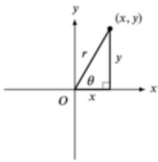
$$\text{Center} = (x_{center}, y_{center})$$

$$\text{Center} = (3, 1)$$

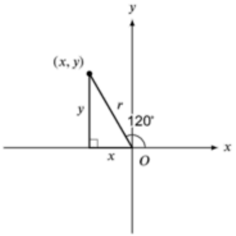
- If a triangle is formed with one vertex at the center of the circle and the other two vertices at points on the circle, then two sides of the triangle are equal to the radius, and thus the triangle is isosceles.

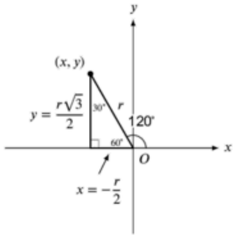
- When we graph a line segment of length r having one end at the origin, extending to a point (x, y), and lying at an angle θ measured relative to the positive x-axis, we can form a triangle as shown below. The horizontal leg has length x, and the vertical leg has length y. The values of x and y can be positive or negative, while r is always positive.





When determining absolute or relative side lengths for such a triangle, be sure to assign the correct sign. In the example below, the short leg lies along the negative x-axis, so its length will be negative. It is essential to have the proper signs for the side lengths when constructing trigonometric ratios for angles represented on the coordinate plane.

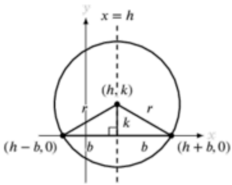


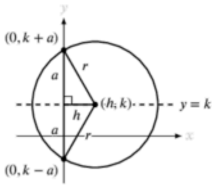


The angle ratios of a right triangle drawn on the coordinate plane are still calculated the same way (remember SOH-CAH-TOA).

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$$

Pairs of points on the circle that have the same y-coordinate will be equidistant from the vertical axis of symmetry, and the distance between such points and the axis of symmetry is simply the difference in the points' x-coordinates and the axis of symmetry's (and thus the center point's) x-coordinate, as seen below, left. With respect to the horizontal axis of symmetry, the same principle applies for pairs of points that have the same x-coordinate, as seen on the right below.





If we're not given the coordinates of the center point, but we have the coordinates of two points on the circle with either the same x- or y-value, **average the x- or y-coordinates of the two points on the circle to find the corresponding coordinate of the center point**.