- The Quadratic Formula can be used to find the solutions of quadratic equations in Standard Form, $0 = ax^2 + bx + c$.

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Completing the Square

$$\begin{aligned} x^2 - 6x + 4 &= 0 \\ x^2 - 6x &= -4 \\ (x-3)^2 &= -4 + 3^2 \\ (x-3)^2 &= -4 + 9 \\ (x-3)^2 &= 5 \\ x - 3 &= \pm\sqrt{5} \\ x &= 3 \pm \sqrt{5} \end{aligned}$$

Quadratic Formula

$$x^2 - 6x + 4 = 0$$

$$x = \frac{-(-6)}{2(1)} \pm \frac{\sqrt{(-6)^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{6}{2} \pm \frac{\sqrt{36 - 16}}{2}$$

$$x = 3 \pm \frac{\sqrt{20}}{2}$$

$$\vdots$$

$$x = 3 \pm \sqrt{5}$$

- If you are given a system of equations consisting of one linear equation and one quadratic equation, or less commonly, two quadratic equations, you can use substitution to collapse the system of two equations into one quadratic equation which you can then solve.

$$\begin{aligned} y &= x^2 \\ y &= 8x + 20 \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} 8x + 20 &= x^2 \\ 0 &= x^2 - 8x - 20 \\ 0 &= (x-10)(x+2) \end{aligned}$$

- When an expression appears multiple times in a quadratic equation, you can probably shortcut the problem by factoring with respect to that expression rather than a single variable.

$$\begin{aligned} (x-4)^2 + 2(x-4) - 15 &= 0 \\ [(x-4) - 3][(x-4) + 5] &= 0 \\ (x-7)(x+1) &= 0 \\ x = 7, \quad x &= -1 \end{aligned}$$

Factoring Perfect Squares / Difference of Squares

- When a binomial expression is squared, the result takes one of the following forms:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

- When a binomial expression $(a + b)$ is multiplied by its conjugate, $(a - b)$, the result is equal to the **difference of squares** $a^2 - b^2$.

$$(a+b)(a-b) = a^2 - b^2$$

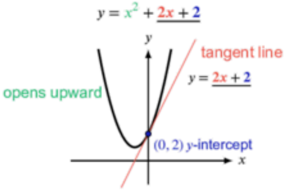
Graphs of Quadratics / Forms of Quadratic Equations

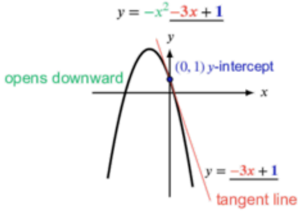
Standard Form

- For quadratics in Standard Form $y = ax^2 + bx + c$, the value of a (the x^2 coefficient) dictates the direction and elongation of the parabola. Positive values cause the graph to open upwards; negative values cause the graph to open downwards.

  The value of b (the x coefficient) is the slope of the tangent line through the y-intercept. The sign of the slope of this tangent line, in conjunction with the sign of the a coefficient, will indicate on which side of the y-axis the vertex lies.

  The value of c (the constant term) is the y-intercept.

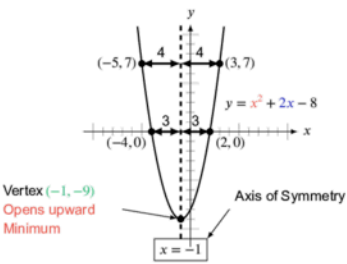




- Parabolas representing quadratic functions of the form $y = f(x)$ are symmetric around a vertical line called the axis of symmetry, which intersects the parabola at one point called the vertex. If the parabola opens upwards (a is positive), the vertex is the lowest point (it has the minimum y-value of any point); if it opens downwards (a is negative), the vertex is the highest point (maximum y-value).

  For a quadratic in Standard Form $y = ax^2 + bx + c$, the x-value of the vertex (and the equation of the axis of symmetry) is

$$x_v = \frac{-b}{2a}$$



$$x_v = \frac{-b}{2a} = \frac{-2}{2(1)} = \frac{-2}{2} = -1$$

Plug the x-value of the vertex into the Standard Form equation in order to easily find the y-value of the vertex. This also allows you to construct the Vertex Form of the parabola from the Standard Form.

$$y_v = (x_v)^2 + 2x_v - 8 = (-1)^2 + 2(-1) - 8 = 1 - 2 - 8 = -9$$

The vertex is $(-1, -9)$ and the value of the leading coefficient is $a = 1$, so the Vertex Form of the equation is $y = (x+1)^2 - 9$.

- To rapidly convert a quadratic equation with an (implied) a-coefficient of 1 from Standard Form to Vertex Form, thus finding the vertex, use the Completing the Square procedure by replacing $x^2 + bx$ with $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$.

Complete the Square: $y = x^2 - 6x - 16$ $\Rightarrow$ $b = -6$, so $\frac{b}{2} = \frac{-6}{2} = -3$ $\Rightarrow$

$$y = \mathbf{x^2 - 6x} - 16$$
$$y = (\mathbf{x} - 3)^2 - (-3)^2 - 16$$
$$y = (\mathbf{x - 3})^2 - \mathbf{9} - 16$$
$$y = (x-3)^2 - 25$$
Vertex $(3, -25)$

If there is a non-1 a-coefficient, replace $ax^2 + bx$ with $a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2$.

Complete the Square: $y = -2x^2 + 4x - 8$ $\Rightarrow$ $a = -2$ and $b = 4$, so $\frac{b}{2a} = \frac{4}{2(-2)} = -1$ $\Rightarrow$

$$y = \mathbf{-2x^2 + 4x} - 8$$
$$y = -2(\mathbf{x} - 1)^2 - [-2(-1)^2] - 8$$
$$y = \mathbf{-2(x-1)^2 + 2} - 8$$
$$y = -2(x-1)^2 - 6$$
Vertex $(1, -6)$

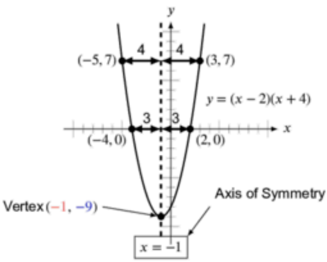
Factored Form

- The Factored Form of a quadratic is

$$y = a(x+p)(x+q)$$

where a, p, and q are constants.

- For quadratics in Factored Form $y = a(x+p)(x+q)$, $-p$ and $-q$ are roots. If $(x - z)$ is a factor of a quadratic, then z is a root of the function and vice versa.

- 

The x-value of the vertex, x_v, is exactly halfway between the two root values and can be found by averaging the two roots (add the roots and divide by 2).

$$x_v = \frac{\text{Sum of roots}}{2} = \frac{-4+2}{2} = \frac{-2}{2} = -1$$

Plug the x-value of the vertex, x_v, into the Factored Form in order to easily find the y-value of the vertex. This allows you to construct the Vertex Form of the parabola from the Factored Form.

$$y_v = (x_v - 2)(x_v + 4) = (-1-2)(-1+4) = (-3)(3) = -9$$

The vertex is $(-1, -9)$ and the value of the leading coefficient is $a = 1$, so the Vertex Form of the equation is $y = (x+1)^2 - 9$.

- To convert from Factored Form to Standard Form, expand the terms and recombine them in order of decreasing degree.

- The graphs of polynomials will "bounce" off of the x-axis when the exponent of a factor is even; they will go through the x-axis if the exponent of a factor is odd.

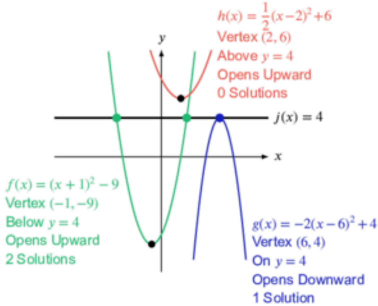
  Even-degree polynomials open upwards when the leading coefficient is positive (downwards when negative). Odd-degree polynomials' y-values go from $-\infty$ to ∞ from left to right when the leading coefficient is positive (vice versa when negative).

Vertex Form

- The Vertex Form of a quadratic is

$$y = a(x-h)^2 + k$$

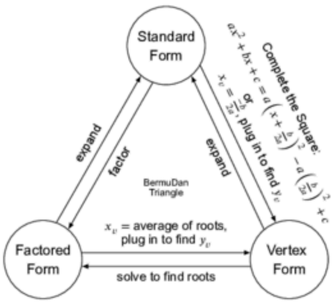
where a, h, and k are constants. The vertex of the graph is (h,k) and the value of a dictates the direction and elongation of the parabola.
The range of a function is the set of all y-values that can be produced by the function. For parabolas that open upwards (the a coefficient is positive), the range is all y-values greater than or equal to the y-value of the vertex. For parabolas that open downwards (the a coefficient is negative), the range is all y-values less than or equal to the y-value of the vertex.



- To convert from Vertex Form to Standard Form, expand the terms and recombine them in order of decreasing degree.

Features of the Forms of Quadratic Equations

- The Standard Form of a quadratic shows its y-intercept as a constant.
- The Factored Form of a quadratic shows its roots as constants.
- The Vertex Form of a quadratic shows its maximum or minimum value as a constant and shows the coordinates of its vertex as a pair of constants.



Number and Type of Zeros of Quadratics

- For quadratics in Standard Form, $y = ax^2 + bx + c$, where a, b, and c are constants, the discriminant

$$b^2 - 4ac$$

  indicates the number of real zeros.

  - When the discriminant is positive, there are 2 real zeros or roots.
  - When the discriminant is equal to 0, there is one real zero or root.
  - When the discriminant is negative, there are no real zeros or roots, but there are 2 complex zeros or roots, and they are the complex conjugates of one another.

- If a Standard Form quadratic is factorable (or is written in Factored Form to begin with), then it has either one (if the quadratic is a perfect square expression) or two real zeros or roots. There is no need to use the discriminant to check for the number of zeros if you can factor the quadratic.
- If a Vertex Form quadratic is given (or you can use Completing the Square to rewrite a Standard Form quadratic in Vertex Form), you can simply visualize or sketch the parabola to determine the number of intersections with the x-axis (these are x-intercepts, which will represent the zeros or roots) because you will know the position of the vertex and whether the parabola opens up or down. Once again, you can forego the use of the discriminant.

Radical & Rational Expressions

Radicals

- When a given equation has an expression in a radical sign, move everything to the other side of the equation before squaring both sides of the equation to eliminate the radical.

$$\sqrt{2x-2}+3=x-2 \quad \Rightarrow \quad \sqrt{2x-2}=x-5 \quad \Rightarrow \quad \left(\sqrt{2x-2}\right)^2=(x-5)^2 \Rightarrow \quad 2x-2=x^2-10x+25$$

- When squaring both sides of an equation, extraneous solutions are often created. Remember that even though both $x=2$ and $x=-2$ are valid solutions to the equation $x^2=4$, when an equation is given to you with an existing square root, the **value of that square root, by definition, is non-negative** (this is also true for all other even roots).
- For questions that ask you to pick the correct solution set for an equation containing a square root, you will need to double check any solutions you find algebraically by plugging those solutions back into the equation, so it may be more efficient to skip the algebra and instead plug in the answer choices.

Which of the values in the set $\{0, 3, 9\}$ are solutions to the equation $\sqrt{2x-2}+3=x-2$?

Check solution $x=0$:

$$\sqrt{2(0)-2}+3=(0)-2$$
$$\sqrt{-2}+3=-2$$
$$✗ \quad \sqrt{-2}\neq -5$$

Check solution $x=3$:

$$\sqrt{2(3)-2}+3=(3)-2$$
$$\sqrt{6-2}+3=1$$
$$\sqrt{4}+3=1$$
$$✗ \quad 2+3\neq 1$$

Check solution $x=9$:

$$\sqrt{2x-2}+3=x-2$$
$$\sqrt{2(9)-2}+3=(9)-2$$
$$\sqrt{18-2}+3=7$$
$$\sqrt{16}+3=7$$
$$✓ \quad 4+3=7$$

Solving algebraically produces two possible solutions, but they need to be tested anyway, so we should go straight to testing values.

$$\sqrt{2x-2}+3=x-2$$
$$\sqrt{2x-2}=x-5$$
$$\left(\sqrt{2x-2}\right)^2=(x-5)^2$$
$$2x-2=x^2-10x+25$$
$$x^2-12x+27=0$$
$$(x-3)(x-9)=0$$
$$x=3,\ x=9$$

Rational Expressions & Remainder Theorem

- Rational expressions consist of one polynomial being divided by another. For some questions, you may need to combine terms through addition or subtraction. To do so, you may need to write terms so that they have common denominators just as you would with any fractions.

$$2+\frac{3}{x+5} \quad \Rightarrow \quad 2\left(\frac{x+5}{x+5}\right)+\frac{3}{x+5} \quad \Rightarrow \quad \frac{2x+10}{x+5}+\frac{3}{x+5} \quad \Rightarrow \quad \frac{2x+13}{x+5}$$

- The value of a rational function is undefined when the denominator is equal to 0. Use factoring if needed to determine what values of x will cause the denominator to be 0.

$$f(x)=\frac{x-1}{x^2+5x+6} \quad \Rightarrow \quad f(x)=\frac{x-1}{(x+2)(x+3)} \quad \Rightarrow \quad \text{Undefined when } x=-2 \text{ and } x=-3$$

- Polynomial Long Division is very rarely necessary (there are always multiple routes through a problem), but you should have a handle on the process just in case.

$$
\begin{array}{rrrr}
 & & x & +5 \\ \cline{2-4}
x+3 \,\big) & x^2 & +8x & +15 \\
 & -(x^2 & +3x) & \Downarrow \\ \cline{2-3}
 & & 5x & +15 \\
 & & -(5x & +15) \\ \cline{3-4}
 & & & 0
\end{array}
\qquad
\begin{array}{rrrr}
 & & x & -4 \\ \cline{2-4}
2x+1 \,\big) & 2x^2 & -7x & -4 \\
 & -(2x^2 & +x) & \Downarrow \\ \cline{2-3}
 & & -8x & -4 \\
 & & -(-8x & -4) \\ \cline{3-4}
 & & & 0
\end{array}
\qquad
\begin{array}{rrrrr}
 & & 3x^2 & +2x & +9 \\ \cline{2-5}
3x-2 \,\big) & 9x^3 & +0x^2 & +23x & -18 \\
 & -(9x^3 & -6x^2) & \Downarrow & \Downarrow \\ \cline{2-3}
 & & 6x^2 & +23x & \Downarrow \\
 & & -(6x^2 & -4x) & \Downarrow \\ \cline{3-4}
 & & & 27x & -18 \\
 & & & -(27x & -18) \\ \cline{4-5}
 & & & & 0
\end{array}
$$

- The Polynomial Remainder Theorem states that when a polynomial $f(x)$ is divided by a binomial $x-r$, the remainder of the division, R, is equal to $f(r)$. Therefore, when $f(r)=0$, there is no remainder, and thus $f(x)$ is divisible by $x-r$.

  For example, if we wanted to check if $f(x)$ is divisible by $x-4$, we could plug 4 into $f(x)$. If $f(4)=0$, then $f(x)$ is divisible by $x-4$.

Imaginary & Complex Numbers

Imaginary and Complex Numbers

- There is an imaginary number i, and $i = \sqrt{-1}$. It follows that $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$; as with all bases, $i^0 = 1$. For powers higher than 4, the pattern repeats for every set of 4. For example, the next four powers of i are as follows:

$$i^5 = i$$
$$i^6 = -1$$
$$i^7 = -i$$
$$i^8 = 1$$

- The complex conjugate of a complex number $a + bi$ is $a - bi$. Multiplying a complex number by its conjugate will eliminate the imaginary part of the complex number, leaving a real number.
- If there is a complex number in the denominator of a fraction, multiply the numerator and denominator by the complex conjugate of the denominator to produce a real number in the denominator.

Ratios, Probability, and Proportions

Ratios and Probability

- A fraction $\frac{a}{b}$ is a form of ratio that can also be expressed with the notation $a:b$, which can be read as "a to b" or "a in b." Sometimes ratios will be expressed as decimal numbers; for example, $\frac{1}{2}$ is equivalent to the decimal 0.5.
- Ratio notation can be used to show the relative values of any number of terms. For example, the side lengths of a triangle might be in a ratio of $3:4:5$.
- Given a ratio of $a:b$, you may need to make fractions for use in solving problems.

  Depending on the problem, it may be beneficial to use a **part-to-part** fraction of $\frac{a}{b}$ or $\frac{b}{a}$, which helps when comparing the amount of one part to the amount of another part. Other times, you might need to use a **part-to-whole** fraction like $\frac{a}{a+b}$ or $\frac{b}{a+b}$, where the denominator is the sum of the parts (the whole), which helps when comparing the amount of one part to the total, or whole, amount.

  For example, a snack mixture contains 2 parts of peanuts and 5 parts of pretzels, so there are a total of 7 parts in the mixture.

  Part-to-Part: the amount of peanuts is $\frac{2}{5}$ the amount of pretzels (the ratio of peanuts to pretzels is $\frac{2}{5}$)

  Part-to-Whole: the amount of peanuts is $\frac{2}{7}$ of the entire mixture (peanuts make up two-sevenths of the mixture)
- Reducing ratios works exactly like reducing fractions. Just as the fraction $\frac{4}{10}$ reduces to $\frac{2}{5}$, so too does the ratio $4:10$ reduce to $2:5$.

  The process works the same for ratios with more than two terms. For example, a ratio of $6:8:10$ can be reduced by dividing all of the terms by 2, resulting in a reduced ratio of $3:4:5$.
- $$\text{Probability} = \frac{\text{Desired Outcomes}}{\text{Possible Outcomes}}$$
- Choose values from tables very carefully, making sure to restrict yourself appropriately based on the selection criteria.

  Start by determining the **reference** total number of events or entities (people, things, etc.), which will act as the **denominator**. **This reference number is often found in an "if" statement in the question prompt**. For example, if a question contains the phrase, "If a plumber from California is chosen at random..." then the denominator in the fraction is the number of plumbers from California.

  The numerator must be smaller than the denominator because we are always looking for a subset of the reference group that was used for the denominator. Completing the example phrase we just started: "If a plumber from California is chosen at random, what is the probability that he will have more than 4 years of experience?" For the numerator, draw only from the pool of people who fit into the reference category—they must be plumbers from California. Then, we need just those plumbers from California *who have more than 4 years of experience*.

Proportions

- Proportions are formed by setting two ratios equal to each other. Proportions are easy ways to solve for values that have constant rates of change (instead of using linear equations).

  For example, if a table manufacturer makes tables of varying sizes but the length to width ratio of their tables is always 5 to 2 $(5:2)$, then you can write the following proportion:

$$\frac{\text{length}}{\text{width}} = \frac{5}{2}$$

  We recommend writing proportions with the unknown in the numerator of the ratio on the left side. For example, the same table manufacturer wants to make a table with a length of 8 feet that conforms to the standard 2 to 5 width to length ratio. We can solve for the width of that table quickly using the following proportion.

$$\frac{\text{width}}{\text{length}} = \frac{2}{5} \quad \Rightarrow \quad \frac{w}{8\,\text{ft}} = \frac{2}{5} \quad \Rightarrow \quad w = \frac{2(8\,\text{ft})}{5} \quad \Rightarrow \quad w = \frac{16}{5}$$

Relative Change in Non-Linear Relationships

- Sometimes relationship problems involve squared or cubed relationships rather than linear relationships. For example, the area of circle is given by the formula $A = \pi r^2$, where A is the area and r is the radius. Doubling the radius does not double the area. Due to the squared relationship, doubling the radius actually quadruples the area of the circle.

$$A_{\text{original}} = \pi r^2 \quad \Rightarrow \quad A_{\text{new}} = \pi(2r)^2 = 4\pi r^2 \quad \Rightarrow \quad A_{\text{new}} = 4A_{\text{original}}$$

