Lines

Slope-Intercept Form Linear Equations

- Slope-Intercept Form Linear Equation:

$$y = mx + b$$

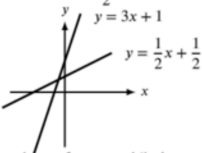
- Total Amount = (Amount per Thing × Number of Things) + Starting Amount

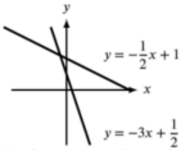
  Example: Jabrill has \$5 (starting amount) and earns \$4 per shirt sold $\Rightarrow$ $j = 4s + 5$
- The four components of a linear equation in the form $y = mx + b$ are the y-value, the x-value, the coefficient m (slope), and the constant b (y-intercept). We can solve for any one of these when we know the other three.
- Slope Formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

- A positive slope means that the y-value increases as the x-value increases. Steeper lines have higher slopes because the y-value goes up more as the x-value increases. A line with a slope of 3 is steeper than a line with a slope of $\frac{1}{2}$.

  A negative slope means that the y-value decreases as the x-value increases. Steeper lines have lower (more negative) slopes because the y-value decreases more as the x-value increases. A line with a slope of -3 is steeper than a line with a slope of $-\frac{1}{2}$.

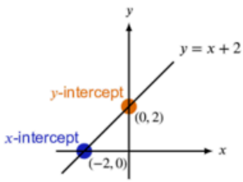




- Horizontal lines have a slope of zero and their equations are of the form $y = c$, where c is a constant (the y-value is the same for all points on the line).

  Vertical lines have an undefined slope and their equations are of the form $x = c$, where c is constant (the x-value is the same for all points on the line).
- The y-intercept is the starting amount or the value of y when the x-value is 0.

  The x-intercept is the value of x when the y-value is equal to 0.



- You can find the equation of any line as long as you know any two points on the line or know the slope and any one point. If you are given the slope and y-intercept, then you already have everything you need to make the equation.

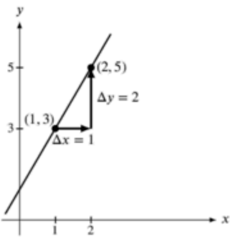
  Step 1: Use the Slope Formula to find the slope from two points: $(1,3)$ and $(2,5)$

  $$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{2-1} = \frac{2}{1} = 2$$

  Step 2: Plug the slope and a point into the Slope-Intercept Form to find the y-intercept: $(1,3)$ and $m = 2$

  $$y = mx + b \quad \Rightarrow \quad 3 = 2(1) + b \quad \Rightarrow \quad 3 = 2 + b \quad \Rightarrow \quad 1 = b \quad \Rightarrow \quad \boxed{y = 2x + 1}$$

- Using up-and-over arrows with graphical representations to find Δy and Δx, or counting differences between coordinate values between rows of tables, is often less error prone than substituting the coordinates of points into the long form of the Slope Formula.



$\Delta x = 1$

x	y
1	3
2	5
3	7
4	9

$\Delta y = 2$

- When asked to interpret the meaning of a coefficient or constant in a Slope-Intercept Form linear equation, first determine which one of the four components of a linear equation it is (the y-value, the slope, the x-value, or the y-intercept).

  The y-intercept is the "starting amount" and is measured in the same units as the y-value.

  The slope is the rate of change of the y-value as the x-value changes and is measured in the units of the y-value divided by the units of the x-value (look for keywords like "per" which indicate division).

Standard Form Linear Equations

- Standard Form Linear Equation:

  $$Ax + By = C$$

- Convert Standard Form to Slope-Intercept Form:

  $$y = \frac{-A}{B}x + \frac{C}{B}$$

- Slope of a Line in Standard Form:

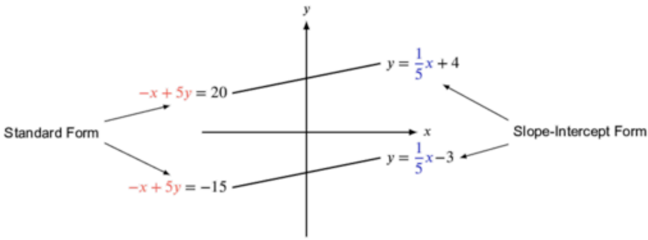
  $$\frac{-A}{B}$$

- y-intercept of a Line in Standard Form:

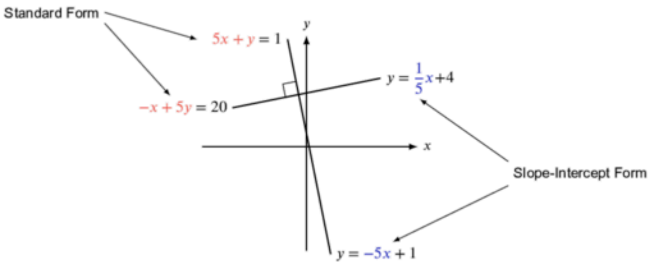
$$\frac{C}{B}$$

Parallel and Perpendicular Lines

- Parallel lines are the same distance apart everywhere and thus they never intersect. Parallel Lines have the same slope but different y-intercepts.
- Parallel lines in Standard Form will have the same x- and y-coefficients (or the ratio of x- and y-coefficients will be the same).



- Perpendicular lines cross at a right angle. The slope of each line in a pair of perpendicular lines is the **negative reciprocal** of the slope of the other line.
- Perpendicular lines in Standard Form will have the x- and y-coefficients swapped, and one of the coefficients will be multiplied by -1 (or the ratio of the x- and y-coefficients will be flipped and multiplied by -1).



Systems of Linear Equations

Substitution

- Solutions to systems of linear equations can be thought of as points where the graphs of the lines intersect. At these intersection points, both lines have the same x- and y- values.

➔ Set expressions for y equal to each other to find the solution point when both equations are already in Slope-Intercept Form. Use substitution when the equations are in different forms, particularly when one variable is already solved for in terms of the other.

$$\left.\begin{array}{l} y = 2x + 1 \\ y = x + 4 \end{array}\right\} \Rightarrow \quad 2x + 1 = -x + 4 \quad \Rightarrow \quad 3x = 3 \quad \Rightarrow \quad x = 1$$

$$\left.\begin{array}{r} x + 4y = 20 \\ x = 4y \end{array}\right\} \Rightarrow \quad (4y) + 4y = 56 \quad \Rightarrow \quad 8y = 56 \quad \Rightarrow \quad y = 7$$

Elimination

➔ Use elimination to solve most systems of linear equations, particularly when both equations are in Standard Form. Multiply one or both equations by numbers that will cause the coefficients of the variable you want to eliminate to cancel out when the equations are combined through addition or subtraction.

Solve for f:

$$\left.\begin{array}{r} f + s = 30 \\ 4f + 6s = 140 \end{array}\right\} \Rightarrow \quad \left.\begin{array}{r} -6(f + s) = -6(30) \\ 4f + 6s = 140 \end{array}\right\} \Rightarrow \quad \begin{array}{rr} & -6f - 6s = -180 \\ + & 4f + 6s = 140 \\ \hline & -2f \quad = -40 \end{array} \quad \Rightarrow \quad f = 20$$

Solve for x:

$$\left.\begin{array}{r} -5x + 2y = 10 \\ 4x + 5y = 25 \end{array}\right\} \Rightarrow \quad \left.\begin{array}{r} -5(-5x + 2y) = -5(10) \\ 2(4x + 5y) = 2(25) \end{array}\right\} \Rightarrow \quad \begin{array}{rr} & 25x - 10y = -50 \\ + & 8x + 10y = 50 \\ \hline & 33x \quad = 0 \end{array} \quad \Rightarrow \quad x = 0$$

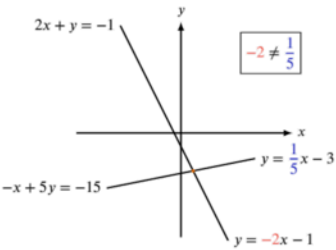
➔ Use combination without eliminating either variable if possible when the problem asks for an expression involving terms with both variables.

Find $x + y$:

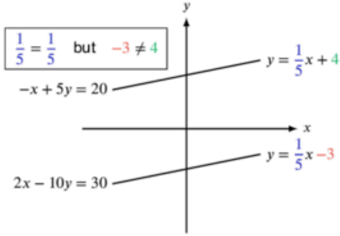
$$\begin{array}{rr} & 2x + 3y = 100 \\ + & 4x + 3y = 380 \\ \hline & 6x + 6y = 480 \end{array} \quad \Rightarrow \quad x + y = 80$$

Number of Solutions to Systems of Linear Equations

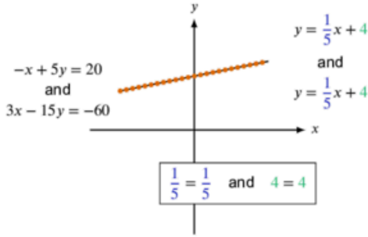
➔ A system of linear equations has **one solution when the two lines are *not* parallel (have different slopes) and thus cross at one point**.



➔ A system of linear equations has **no solutions when the lines are parallel** (have the same slopes and different y-intercepts), because the lines do not intersect.



➔ A system of linear equations has **infinite solutions when the two lines are exactly the same**. The **slopes and y-intercepts must be the same**.



➔ When asked to find unknown coefficient values in systems of linear equations in Standard Form that have no solutions or infinitely many solutions, use proportions of the ratios of the coefficients and constants to solve for unknown values.

$$\text{No Solutions:} \quad \begin{aligned} 12x - 4y &= 2 \\ ax + 2y &= 7 \end{aligned} \quad \Rightarrow \quad \frac{a}{12} = \frac{2}{-4} \quad \Rightarrow \quad a = 12\left(\frac{-1}{2}\right) \quad \Rightarrow \quad a = -6$$

$$\infty \text{ Solutions:} \quad \begin{aligned} 4x + 6y &= 12 \\ -2x - 3y &= b \end{aligned} \quad \Rightarrow \quad \frac{b}{12} = \frac{-3}{6} \quad \Rightarrow \quad b = 12\left(\frac{-1}{2}\right) \quad \Rightarrow \quad b = -6$$

Alternatively, multiply one or both equations by scale factors so that any known coefficients or constants in the same position can be matched between the two equations.

$$\text{No Solutions:} \quad \begin{aligned} 8x - 2y &= 1 \\ ax + 3y &= 3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} -3(8x - 2y) &= -3(1) \\ 2(ax + 3y) &= 2(3) \end{aligned} \quad \Rightarrow \quad \begin{aligned} -24x + 6y &= -3 \\ 2ax + 6y &= 6 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 2a &= -24 \\ a &= -12 \end{aligned}$$

$$\infty \text{ Solutions:} \quad \begin{aligned} 3x - 9y &= 12 \\ ax + by &= 4 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 3x - 9y &= 12 \\ 3(ax + by) &= 3(4) \end{aligned} \quad \Rightarrow \quad \begin{aligned} 3x - 9y &= 12 \\ 3ax + 3by &= 12 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 3a &= 3 \\ a &= 1 \end{aligned} \quad \text{and} \quad \begin{aligned} 3b &= -9 \\ b &= -3 \end{aligned}$$

Linear Inequalities & Absolute Value

Linear Inequalities

- The pointy or small end of the inequality sign is the lesser side of the inequality; the open or big end of the inequality sign is the greater side of the inequality.

Name	Symbol	Usage
Less Than	$<$	The value on the left is less than the value on the right. The inequality $x < 5$ means that x can be any value from $-\infty$ up to, but **not** including, 5.
Less Than Or Equal To	$\leq$	The value on the left is less than or equal to (no more than) the value on the right. For example, $x \leq 5$, means that x can be any value from $-\infty$ up to 5, including 5 itself.
Greater Than	$>$	The value on the left is greater than the value on the right . The inequality $x > 5$ means that x can be any value greater than 5 up to ∞.
Greater Than Or Equal To	$\geq$	The value on the left is greater than or equal to (no less than) the value on the right. For example, $x \geq 5$ means that x can be any value from 5 (including 5 itself) up to ∞.
Absolute Value	$\lvert a \rvert$	Produces a non-negative value indicating how far a number is from 0. ($\lvert -5 \rvert = 5$)

- Solve inequalities like you solve equations, but remember to flip the sign if you multiply or divide by a negative number.

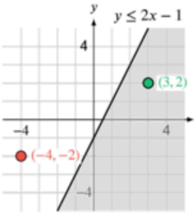
$$-8x + 3 \leq 59 \quad \Rightarrow \quad -8x \leq 56 \quad \Rightarrow \quad x \geq -7$$

- In real world problems, correct answers often have to be integers, particularly if the problem involves items that cannot be divided into fractional amounts (books, shirts, etc.).
- Compound inequalities can be solved in one step as long as you are careful about flipping both inequality signs if you multiply or divide by negative numbers. Just make sure to apply the same operation to all three parts of the compound inequality.

$$2 < -3x + 2 < 8 \quad \Rightarrow \quad 0 < -3x < 6 \quad \Rightarrow \quad \frac{0}{-3} > x > \frac{6}{-3} \quad \Rightarrow \quad 0 > x > -2$$

- When graphing linear inequalities in the xy-plane, use a solid line if there is a $\leq$ or $\geq$ sign in the inequality (because points on the line **are** solutions to the inequality), and use a dashed line if there is a $<$ or $>$ sign in the inequality (because points on the line **are NOT** solutions to the inequality). Shade **above** the line to show the solution region when the y-variable is **greater than** the linear expression. Shade **below** the line to show the solution region when the y-variable is **less than** the linear expression.

  Plug in the x- and y-coordinates of a point to the inequality to see if a point falls in the shaded solution region or is on a solid boundary line.



$$y \leq 2x - 1$$

Test $(3, 2)$

$2 \leq 2(3) - 1$

$2 \leq 6 - 1$

✓ $2 < 5$

Test $(-4, -2)$

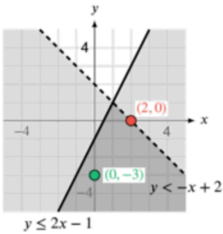
$-2 \leq 2(-4) - 1$

$-2 \leq -8 - 1$

✗ $-2 \nleq -9$

Systems of Linear Inequalities

- Solutions to a system of linear inequalities must satisfy all the inequalities in the system. If the inequalities are graphed on the same xy-plane, the solutions are located within the overlapping shaded regions and on any solid lines bounding those regions.



$$y \leq 2x - 1$$
$$y < -x + 2$$

Test $(0, -3)$

$-3 \leq 2(0) - 1$

$-3 < -(0) + 2$

$\Downarrow$

✓ $\mathbf{-3 \leq -1}$

✓ $\mathbf{-3 < 2}$

Test $(2, 0)$

$0 \leq 2(2) - 1$

$0 < -(2) + 2$

$\Downarrow$

$0 \leq 4 - 1$

$0 < -2 + 2$

$\Downarrow$

✓ $\mathbf{0 \leq 3}$

✗ $\mathbf{0 \nless 0}$

Absolute Value

- You can solve absolute value equations by making two equations: one where the expression in the absolute value bars is set equal to the original value on the other side of the equation, and one where the expression is set equal to the negative of that value. For example, $|x| = 5$ yields both $x = 5$ and $x = -5$.

  Note that while the expression *within* the absolute value bars can have any value, positive or negative, **the absolute value of the expression can never be negative, by definition**. For example, while $|x| = 5$ yields two possible solutions ($x = 5$ and $x = -5$), the equation $|x| = -5$ yields **zero solutions** because the absolute value of an expression can never be negative.

- $|x + a| = b \quad \Rightarrow \quad \begin{cases} x + a = b \\ x + a = -b \end{cases}$

- $|x + a| > b \quad \Rightarrow \quad \begin{cases} x + a > b \\ x + a < -b \end{cases}$

- $|x + a| < b \quad \Rightarrow \quad \begin{cases} x + a < b \\ x + a > -b \end{cases} \Rightarrow \quad -b < x + a < b$

Exponents

Exponent Rules

- Exponential terms consist of a base a and exponent b and are in the form a^b.
- Exponents tell you how many factors of the base should be multiplied by each other. For example, $2^5 = 2 \times 2 \times 2 \times 2 \times 2$ (there are 5 factors of 2 multiplied by each other).
- $a^b a^c = a^{b+c}$
- $\left(a^b\right)^c = a^{bc}$
- $\dfrac{a^b}{a^c} = a^{b-c}$
- $a^{-b} = \dfrac{1}{a^b}$ and $\dfrac{1}{a^{-b}} = a^b$
- $(ab)^c = a^c b^c$ BUT WATCH OUT FOR THIS MISTAKE: $(a+b)^c \neq a^c + b^c$
- $a^{\frac{b}{c}} = \sqrt[c]{a^b} = \left(\sqrt[c]{a}\right)^b$ Note that the rule will hold as long as the base, a, is non-negative. Rewrite radical expressions as terms with fractional exponents.

Methods of Solving Exponent Equations

- You can solve exponential equations by raising both sides to a power such that the exponents of the variable term will multiply to the correct power (the power you are looking for) on one side of the equation. For example if you are given the equation $x^{\frac{1}{5}} = 2$, you can solve for x^1 by raising both sides of the equation to the fifth power:

$$\left(x^{\frac{1}{5}}\right)^5 = 2^5 \quad \Rightarrow \quad x = 32$$

- You can write relatively large bases in terms of smaller bases in order to rewrite exponential terms with fractional exponents (without using a calculator) in order to match answer choices.

  For example, if we wanted to simplify the term $9^{\frac{3}{4}}$, we could rewrite the 9 as 3^2, allowing us to break the exponent up and rewrite $9^{\frac{3}{4}}$ as a radical expression with a base of 3.

$$9^{\frac{3}{4}} = \left(3^2\right)^{\frac{3}{4}} = 3^{2\left(\frac{3}{4}\right)} = 3^{\frac{3}{2}} = 3^{\left(1+\frac{1}{2}\right)} = 3 \cdot 3^{\frac{1}{2}} = 3\sqrt{3}$$

  It is also advantageous to write terms in another base whenever you have two different base numbers that share a common factor.

  For example, if we are given the equation $27^x \cdot 3^{4y} = 3^5$ and asked to solve for $3x + 4y$, we can combine the terms in the expression on the left side if we rewrite 27 as a base of 3 raised to the third power, allowing us to apply exponent rules to combine the terms.

$$27^x \cdot 3^{4y} = 3^5 \quad \Rightarrow \quad \left(3^3\right)^x \cdot 3^{4y} = 3^5 \quad \Rightarrow \quad 3^{3x} \cdot 3^{4y} = 3^5 \quad \Rightarrow \quad 3^{3x+4y} = 3^5$$

  Changing the base was useful in this example because it allowed us to write both sides of the equation as 3 raised to a power, which allows us to very easily see that $3x + 4y$ must be equal to 5 (since 3 to one power can't be equal to 3 to a different power).

Quadratics & Other Polynomials

Standard Form Polynomials

- The Standard Form for a univariate (single variable) polynomial puts the terms in order from highest to lowest power, so the Standard Form of a quadratic (second-degree polynomial) equation is

$$y = ax^2 + bx + c$$

where a, b, and c are constants.

- When adding or subtracting expressions, group and combine like terms of the same variable (same base and exponent).

$$(3x + 1 - 2y^2) - (2x + 4 - 5y^2)$$
$$3x - 2x + 1 - 4 - 2y^2 + 5y^2$$
$$x - 3 + 3y^2$$

- When multiplying expressions, distribute all terms in the first expression to all terms in the second expression.

$$(2x + 3)(x + 2) = 2x(x + 2) + 3(x + 2)$$
$$(2x + 3)(x + 2) = 2x(x) + 2x(2) + 3(x) + 3(2)$$
$$(2x + 3)(x + 2) = 2x^2 + 4x + 3x + 6$$
$$(2x + 3)(x + 2) = 2x^2 + 7x + 6$$

Solving Polynomial Equations

- If the two sides of an equation are polynomials written in Standard Form, and you are told that the equation is true for all values of the variable (that is, the polynomials are **equivalent**), you can match the coefficients of corresponding terms to find the values of any unknown coefficients.

$$-2x(5x + 1) + 6(5x + 1) = ax^2 + bx + c$$
$$-10x^2 - 2x + 30x + 6 = ax^2 + bx + c$$
$$-10x^2 + 28x + 6 = ax^2 + bx + c \Rightarrow$$

$a = -10$, $b = 28$, and $c = 6$

Factoring

- When factoring a quadratic expression in Standard Form where the coefficient of the x^2 term is 1, $x^2 + bx + c = 0$, we need to find two numbers, p and q, such that $p + q = b$ (the sum of the numbers is b) and $pq = c$ (the product of the numbers is c). The solutions to the equation are $-p$ and $-q$, and we can rewrite the expression in Factored Form. One of the factors is $(x + p)$ and the other factor is $(x + q)$.

$$(x + p)(x + q) = 0$$

For example, to find the factors of the quadratic expression in the equation $y = x^2 + 3x + 2$, we need to find two numbers that add to 3 and multiply to 2. The numbers 1 and 2 add to 3 (we know that $1 + 2 = 3$) and multiply to 2 (we know that $1(2) = 2$), so the factors are $(x + 1)$ and $(x + 2)$. The solutions are thus -1 and -2.

$$y = x^2 + 3x + 2$$
$$y = (x + 1)(x + 2)$$

- The sum of the solutions of a quadratic equation in Standard Form, $ax^2 + bx + c = 0$, is equal to

$$\frac{-b}{a}$$

- The product of the solutions of a quadratic equation in Standard Form, $ax^2 + bx + c = 0$, is equal to

$$\frac{c}{a}$$

Other Methods of Finding the Roots of Quadratics

- When the coefficient of the x^2 term is NOT 1 (when $a \neq 1$), first try dividing all of the terms by a to see if the resulting expression is easily factorable.

$$3x^2 - 12x - 15 = 0$$

$$\frac{3x^2 - 12x - 15}{3} = \frac{0}{3}$$

$$x^2 - 4x - 5 = 0$$

$$(x-5)(x+1) = 0$$

$$x = 5\,, x = -1$$

- Sometimes, when the coefficient of the x^2 term is NOT 1 (when $a \neq 1$), you can use the Product Sum (ac) Method, which is a generalized form of factoring.
  When factoring a quadratic expression in Standard Form $ax^2 + bx + c = 0$ using the Product Sum (ac) Method, we need to find two numbers, r and s, such that $r + s = b$ (the sum of the numbers is b) and $rs = ac$ (the product of the numbers is ac). The solutions to the equation are $\frac{-r}{a}$ and $\frac{-s}{a}$, and we can rewrite the expression in Factored Form. One of the factors is $\left(x + \frac{r}{a}\right)$ and the other factor is $\left(x + \frac{s}{a}\right)$.

$$a\left(x + \frac{r}{a}\right)\left(x + \frac{s}{a}\right) = 0$$

$$4x^2 - x - 3 = 0$$

Product: $ac = 4(-3) = -12$
Sum: $b = -1$
The numbers r and s whose sum is -1 and whose product is -12 are 3 and -4, so $r = 3$ and $s = -4$.
The zeros are $x = \frac{-r}{a} = \frac{-3}{4}$ and $x = \frac{-s}{a} = \frac{-(-4)}{4} = 1$.
The factors are $\left(x + \frac{3}{4}\right)$ and $(x - 1)$.

- If you cannot find numbers that satisfy the conditions for factoring, you can always use Completing the Square:

1. Divide both sides of the equation by the a coefficient to eliminate the coefficient of the x^2 term.
2. Move the constant to other side to isolate the x terms.
3. Replace the left side of the equation with $\left(x + \frac{b}{2}\right)^2$ (where b is the coefficient of the x term) and balance the extra constant on the other side by adding $\left(\frac{b}{2}\right)^2$ to the right side of the equation, pre-calculating $\frac{b}{2}$ so you don't have to do that twice.
4. Take the square root of both sides, being careful to include both positive and negative square roots.
5. Move the constant to right side of the equation to isolate and solve for x.
6. Enumerate the solutions.

- The Quadratic Formula can be used to find the solutions of quadratic equations in Standard Form, $0 = ax^2 + bx + c$.

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Completing the Square

$$\begin{aligned} x^2 - 6x + 4 &= 0 \\ x^2 - 6x &= -4 \\ (x-3)^2 &= -4 + 3^2 \\ (x-3)^2 &= -4 + 9 \\ (x-3)^2 &= 5 \\ x - 3 &= \pm\sqrt{5} \\ x &= 3 \pm \sqrt{5} \end{aligned}$$

Quadratic Formula

$$x^2 - 6x + 4 = 0$$

$$x = \frac{-(-6)}{2(1)} \pm \frac{\sqrt{(-6)^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{6}{2} \pm \frac{\sqrt{36 - 16}}{2}$$

$$x = 3 \pm \frac{\sqrt{20}}{2}$$

$$\vdots$$

$$x = 3 \pm \sqrt{5}$$

- If you are given a system of equations consisting of one linear equation and one quadratic equation, or less commonly, two quadratic equations, you can use substitution to collapse the system of two equations into one quadratic equation which you can then solve.

$$\begin{aligned} y &= x^2 \\ y &= 8x + 20 \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} 8x + 20 &= x^2 \\ 0 &= x^2 - 8x - 20 \\ 0 &= (x - 10)(x + 2) \end{aligned}$$

- When an expression appears multiple times in a quadratic equation, you can probably shortcut the problem by factoring with respect to that expression rather than a single variable.

$$\begin{aligned} (x-4)^2 + 2(x-4) - 15 &= 0 \\ [(x-4) - 3][(x-4) + 5] &= 0 \\ (x-7)(x+1) &= 0 \\ x = 7, \quad x &= -1 \end{aligned}$$

Factoring Perfect Squares / Difference of Squares

- When a binomial expression is squared, the result takes one of the following forms:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

- When a binomial expression $(a + b)$ is multiplied by its conjugate, $(a - b)$, the result is equal to the **difference of squares** $a^2 - b^2$.

$$(a+b)(a-b) = a^2 - b^2$$

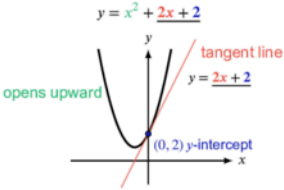
Graphs of Quadratics / Forms of Quadratic Equations

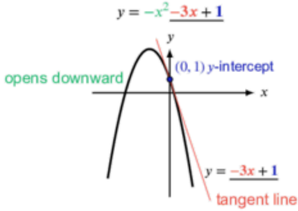
Standard Form

- For quadratics in Standard Form $y = ax^2 + bx + c$, the value of a (the x^2 coefficient) dictates the direction and elongation of the parabola. Positive values cause the graph to open upwards; negative values cause the graph to open downwards.

  The value of b (the x coefficient) is the slope of the tangent line through the y-intercept. The sign of the slope of this tangent line, in conjunction with the sign of the a coefficient, will indicate on which side of the y-axis the vertex lies.

  The value of c (the constant term) is the y-intercept.

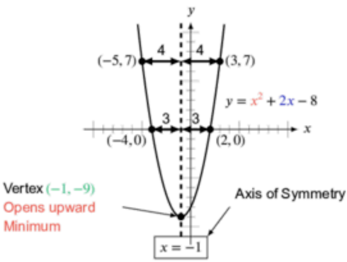




- Parabolas representing quadratic functions of the form $y = f(x)$ are symmetric around a vertical line called the axis of symmetry, which intersects the parabola at one point called the vertex. If the parabola opens upwards (a is positive), the vertex is the lowest point (it has the minimum y-value of any point); if it opens downwards (a is negative), the vertex is the highest point (maximum y-value).

  For a quadratic in Standard Form $y = ax^2 + bx + c$, the x-value of the vertex (and the equation of the axis of symmetry) is

$$x_v = \frac{-b}{2a}$$



$$x_v = \frac{-b}{2a} = \frac{-2}{2(1)} = \frac{-2}{2} = -1$$

Plug the x-value of the vertex into the Standard Form equation in order to easily find the y-value of the vertex. This also allows you to construct the Vertex Form of the parabola from the Standard Form.

$$y_v = (x_v)^2 + 2x_v - 8 = (-1)^2 + 2(-1) - 8 = 1 - 2 - 8 = -9$$

The vertex is $(-1, -9)$ and the value of the leading coefficient is $a = 1$, so the Vertex Form of the equation is $y = (x+1)^2 - 9$.

- To rapidly convert a quadratic equation with an (implied) a-coefficient of 1 from Standard Form to Vertex Form, thus finding the vertex, use the Completing the Square procedure by replacing $x^2 + bx$ with $\left(x+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$.

Complete the Square: $y = x^2 - 6x - 16$ $\Rightarrow$ $b = -6$, so $\frac{b}{2} = \frac{-6}{2} = -3$ $\Rightarrow$

$$y = \mathbf{x^2 - 6x} - 16$$
$$y = (\mathbf{x-3})^2 - (-3)^2 - 16$$
$$y = (\mathbf{x-3})^2 - \mathbf{9} - 16$$
$$y = (x-3)^2 - 25$$

Vertex $(3, -25)$

If there is a non-1 a-coefficient, replace $ax^2 + bx$ with $a\left(x+\frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2$.

Complete the Square: $y = -2x^2 + 4x - 8$ $\Rightarrow$ $a = -2$ and $b = 4$, so $\frac{b}{2a} = \frac{4}{2(-2)} = -1$ $\Rightarrow$

$$y = \mathbf{-2x^2 + 4x} - 8$$
$$y = -2(\mathbf{x-1})^2 - [-2(-1)^2] - 8$$
$$y = -2(\mathbf{x-1})^2 + \mathbf{2} - 8$$
$$y = -2(x-1)^2 - 6$$

Vertex $(1, -6)$

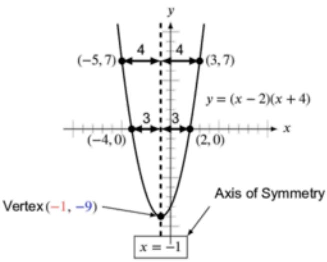
Factored Form

- The Factored Form of a quadratic is

$$y = a(x+p)(x+q)$$

where a, p, and q are constants.

- For quadratics in Factored Form $y = a(x+p)(x+q)$, $-p$ and $-q$ are roots. If $(x - z)$ is a factor of a quadratic, then z is a root of the function and vice versa.

- 

The x-value of the vertex, x_v, is exactly halfway between the two root values and can be found by averaging the two roots (add the roots and divide by 2).

$$x_v = \frac{\text{Sum of roots}}{2} = \frac{-4+2}{2} = \frac{-2}{2} = -1$$

Plug the x-value of the vertex, x_v, into the Factored Form in order to easily find the y-value of the vertex. This allows you to construct the Vertex Form of the parabola from the Factored Form.

$$y_v = (x_v - 2)(x_v + 4) = (-1-2)(-1+4) = (-3)(3) = -9$$

The vertex is $(-1, -9)$ and the value of the leading coefficient is $a = 1$, so the Vertex Form of the equation is $y = (x+1)^2 - 9$.

- To convert from Factored Form to Standard Form, expand the terms and recombine them in order of decreasing degree.

- The graphs of polynomials will "bounce" off of the x-axis when the exponent of a factor is even; they will go through the x-axis if the exponent of a factor is odd.

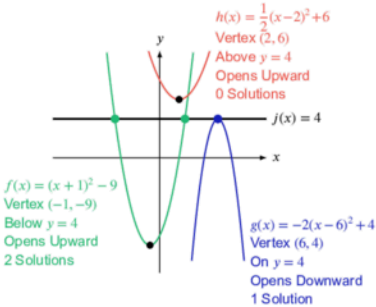
  Even-degree polynomials open upwards when the leading coefficient is positive (downwards when negative). Odd-degree polynomials' y-values go from $-\infty$ to ∞ from left to right when the leading coefficient is positive (vice versa when negative).

Vertex Form

- The Vertex Form of a quadratic is

$$y = a(x-h)^2 + k$$

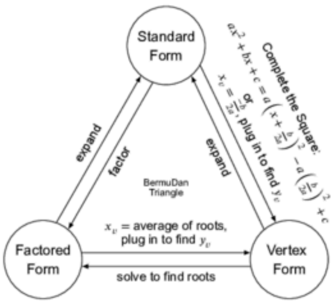
where a, h, and k are constants. The vertex of the graph is (h, k) and the value of a dictates the direction and elongation of the parabola.
The range of a function is the set of all y-values that can be produced by the function. For parabolas that open upwards (the a coefficient is positive), the range is all y-values greater than or equal to the y-value of the vertex. For parabolas that open downwards (the a coefficient is negative), the range is all y-values less than or equal to the y-value of the vertex.



- To convert from Vertex Form to Standard Form, expand the terms and recombine them in order of decreasing degree.

Features of the Forms of Quadratic Equations

- The Standard Form of a quadratic shows its y-intercept as a constant.
- The Factored Form of a quadratic shows its roots as constants.
- The Vertex Form of a quadratic shows its maximum or minimum value as a constant and shows the coordinates of its vertex as a pair of constants.



Number and Type of Zeros of Quadratics

- For quadratics in Standard Form, $y = ax^2 + bx + c$, where a, b, and c are constants, the discriminant

$$b^2 - 4ac$$

  indicates the number of real zeros.

  - When the discriminant is positive, there are 2 real zeros or roots.
  - When the discriminant is equal to 0, there is one real zero or root.
  - When the discriminant is negative, there are no real zeros or roots, but there are 2 complex zeros or roots, and they are the complex conjugates of one another.

- If a Standard Form quadratic is factorable (or is written in Factored Form to begin with), then it has either one (if the quadratic is a perfect square expression) or two real zeros or roots. There is no need to use the discriminant to check for the number of zeros if you can factor the quadratic.

- If a Vertex Form quadratic is given (or you can use Completing the Square to rewrite a Standard Form quadratic in Vertex Form), you can simply visualize or sketch the parabola to determine the number of intersections with the x-axis (these are x-intercepts, which will represent the zeros or roots) because you will know the position of the vertex and whether the parabola opens up or down. Once again, you can forego the use of the discriminant.

Radical & Rational Expressions

Radicals

- When a given equation has an expression in a radical sign, move everything to the other side of the equation before squaring both sides of the equation to eliminate the radical.

$$\sqrt{2x-2}+3=x-2 \quad \Rightarrow \quad \sqrt{2x-2}=x-5 \quad \Rightarrow \quad \left(\sqrt{2x-2}\right)^2=(x-5)^2 \Rightarrow \quad 2x-2=x^2-10x+25$$

- When squaring both sides of an equation, extraneous solutions are often created. Remember that even though both $x=2$ and $x=-2$ are valid solutions to the equation $x^2=4$, when an equation is given to you with an existing square root, the **value of that square root, by definition, is non-negative** (this is also true for all other even roots).
- For questions that ask you to pick the correct solution set for an equation containing a square root, you will need to double check any solutions you find algebraically by plugging those solutions back into the equation, so it may be more efficient to skip the algebra and instead plug in the answer choices.

Which of the values in the set $\{0, 3, 9\}$ are solutions to the equation $\sqrt{2x-2}+3=x-2$?

Check solution $x=0$:

$$\sqrt{2(0)-2}+3=(0)-2$$
$$\sqrt{-2}+3=-2$$
$$✗ \quad \sqrt{-2}\neq-5$$

Check solution $x=3$:

$$\sqrt{2(3)-2}+3=(3)-2$$
$$\sqrt{6-2}+3=1$$
$$\sqrt{4}+3=1$$
$$✗ \quad 2+3\neq1$$

Check solution $x=9$:

$$\sqrt{2x-2}+3=x-2$$
$$\sqrt{2(9)-2}+3=(9)-2$$
$$\sqrt{18-2}+3=7$$
$$\sqrt{16}+3=7$$
$$✓ \quad 4+3=7$$

Solving algebraically produces two possible solutions, but they need to be tested anyway, so we should go straight to testing values.

$$\sqrt{2x-2}+3=x-2$$
$$\sqrt{2x-2}=x-5$$
$$\left(\sqrt{2x-2}\right)^2=(x-5)^2$$
$$2x-2=x^2-10x+25$$
$$x^2-12x+27=0$$
$$(x-3)(x-9)=0$$
$$x=3,\ x=9$$

Rational Expressions & Remainder Theorem

- Rational expressions consist of one polynomial being divided by another. For some questions, you may need to combine terms through addition or subtraction. To do so, you may need to write terms so that they have common denominators just as you would with any fractions.

$$2+\frac{3}{x+5} \quad \Rightarrow \quad 2\left(\frac{x+5}{x+5}\right)+\frac{3}{x+5} \quad \Rightarrow \quad \frac{2x+10}{x+5}+\frac{3}{x+5} \quad \Rightarrow \quad \frac{2x+13}{x+5}$$

- The value of a rational function is undefined when the denominator is equal to 0. Use factoring if needed to determine what values of x will cause the denominator to be 0.

$f(x)=\dfrac{x-1}{x^2+5x+6} \quad \Rightarrow \quad f(x)=\dfrac{x-1}{(x+2)(x+3)} \quad \Rightarrow$ Undefined when $x=-2$ and $x=-3$

- Polynomial Long Division is very rarely necessary (there are always multiple routes through a problem), but you should have a handle on the process just in case.

$$
\begin{array}{r|rrr}
 & x & +5 & \\
\hline
x+3 & x^2 & +8x & +15 \\
 & -(x^2 & +3x) & \Downarrow \\
\hline
 & & 5x & +15 \\
 & & -(5x & +15) \\
\hline
 & & & 0
\end{array}
\qquad
\begin{array}{r|rrr}
 & & x & -4 \\
\hline
2x+1 & 2x^2 & -7x & -4 \\
 & -(2x^2 & +x) & \Downarrow \\
\hline
 & & -8x & -4 \\
 & & -(-8x & -4) \\
\hline
 & & & 0
\end{array}
\qquad
\begin{array}{r|rrrr}
 & 3x^2 & +2x & +9 & \\
\hline
3x-2 & 9x^3 & +0x^2 & +23x & -18 \\
 & -(9x^3 & -6x^2) & \Downarrow & \Downarrow \\
\hline
 & & 6x^2 & +23x & \Downarrow \\
 & & -(6x^2 & -4x) & \Downarrow \\
\hline
 & & & 27x & -18 \\
 & & & -(27x & -18) \\
\hline
 & & & & 0
\end{array}
$$

- The Polynomial Remainder Theorem states that when a polynomial $f(x)$ is divided by a binomial $x-r$, the remainder of the division, R, is equal to $f(r)$. Therefore, when $f(r)=0$, there is no remainder, and thus $f(x)$ is divisible by $x-r$.

  For example, if we wanted to check if $f(x)$ is divisible by $x-4$, we could plug 4 into $f(x)$. If $f(4)=0$, then $f(x)$ is divisible by $x-4$.

Imaginary & Complex Numbers

Imaginary and Complex Numbers

- There is an imaginary number i, and $i = \sqrt{-1}$. It follows that $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$; as with all bases, $i^0 = 1$. For powers higher than 4, the pattern repeats for every set of 4. For example, the next four powers of i are as follows:

$$i^5 = i$$
$$i^6 = -1$$
$$i^7 = -i$$
$$i^8 = 1$$

- The complex conjugate of a complex number $a + bi$ is $a - bi$. Multiplying a complex number by its conjugate will eliminate the imaginary part of the complex number, leaving a real number.
- If there is a complex number in the denominator of a fraction, multiply the numerator and denominator by the complex conjugate of the denominator to produce a real number in the denominator.

Ratios, Probability, and Proportions

Ratios and Probability

- A fraction $\frac{a}{b}$ is a form of ratio that can also be expressed with the notation $a:b$, which can be read as "a to b" or "a in b." Sometimes ratios will be expressed as decimal numbers; for example, $\frac{1}{2}$ is equivalent to the decimal 0.5.
- Ratio notation can be used to show the relative values of any number of terms. For example, the side lengths of a triangle might be in a ratio of $3:4:5$.
- Given a ratio of $a:b$, you may need to make fractions for use in solving problems.

  Depending on the problem, it may be beneficial to use a **part-to-part** fraction of $\frac{a}{b}$ or $\frac{b}{a}$, which helps when comparing the amount of one part to the amount of another part. Other times, you might need to use a **part-to-whole** fraction like $\frac{a}{a+b}$ or $\frac{b}{a+b}$, where the denominator is the sum of the parts (the whole), which helps when comparing the amount of one part to the total, or whole, amount.

  For example, a snack mixture contains 2 parts of peanuts and 5 parts of pretzels, so there are a total of 7 parts in the mixture.

  Part-to-Part: the amount of peanuts is $\frac{2}{5}$ the amount of pretzels (the ratio of peanuts to pretzels is $\frac{2}{5}$)

  Part-to-Whole: the amount of peanuts is $\frac{2}{7}$ of the entire mixture (peanuts make up two-sevenths of the mixture)
- Reducing ratios works exactly like reducing fractions. Just as the fraction $\frac{4}{10}$ reduces to $\frac{2}{5}$, so too does the ratio $4:10$ reduce to $2:5$.

  The process works the same for ratios with more than two terms. For example, a ratio of $6:8:10$ can be reduced by dividing all of the terms by 2, resulting in a reduced ratio of $3:4:5$.
- $$\text{Probability} = \frac{\text{Desired Outcomes}}{\text{Possible Outcomes}}$$
- Choose values from tables very carefully, making sure to restrict yourself appropriately based on the selection criteria.

  Start by determining the **reference** total number of events or entities (people, things, etc.), which will act as the **denominator**. **This reference number is often found in an "if" statement in the question prompt**. For example, if a question contains the phrase, "If a plumber from California is chosen at random..." then the denominator in the fraction is the number of plumbers from California.

  The numerator must be smaller than the denominator because we are always looking for a subset of the reference group that was used for the denominator. Completing the example phrase we just started: "If a plumber from California is chosen at random, what is the probability that he will have more than 4 years of experience?" For the numerator, draw only from the pool of people who fit into the reference category—they must be plumbers from California. Then, we need just those plumbers from California *who have more than 4 years of experience*.

Proportions

- Proportions are formed by setting two ratios equal to each other. Proportions are easy ways to solve for values that have constant rates of change (instead of using linear equations).

  For example, if a table manufacturer makes tables of varying sizes but the length to width ratio of their tables is always 5 to 2 ($5:2$), then you can write the following proportion:

$$\frac{\text{length}}{\text{width}} = \frac{5}{2}$$

  We recommend writing proportions with the unknown in the numerator of the ratio on the left side. For example, the same table manufacturer wants to make a table with a length of 8 feet that conforms to the standard 2 to 5 width to length ratio. We can solve for the width of that table quickly using the following proportion.

$$\frac{\text{width}}{\text{length}} = \frac{2}{5} \quad \Rightarrow \quad \frac{w}{8\,\text{ft}} = \frac{2}{5} \quad \Rightarrow \quad w = \frac{2(8\,\text{ft})}{5} \quad \Rightarrow \quad w = \frac{16}{5}$$

Relative Change in Non-Linear Relationships

- Sometimes relationship problems involve squared or cubed relationships rather than linear relationships. For example, the area of circle is given by the formula $A = \pi r^2$, where A is the area and r is the radius. Doubling the radius does not double the area. Due to the squared relationship, doubling the radius actually quadruples the area of the circle.

$$A_{\text{original}} = \pi r^2 \quad \Rightarrow \quad A_{\text{new}} = \pi(2r)^2 = 4\pi r^2 \quad \Rightarrow \quad A_{\text{new}} = 4A_{\text{original}}$$