NORTHWESTERN UNIVERSITY
Department of Electrical and Computer Engineering

ECE 379 – Lecture 1

FIELDS AND WAVES

1.1 Introduction

As the title "Lasers and Coherent Optics" indicates, in this course we will learn the basic principles of laser operation and properties of their coherent outputs. We will do so in a manner that does not invoke quantum mechanics and, instead, draws heavily on an electrical engineer's background, namely, Fourier transforms and linear systems theory. In the first half, we will study the simplified theory of a generic laser, deferring particular laser systems to other courses, and in the latter half we will come to grips with coherent beam propagation, diffraction, imaging, and optical signal processing.

Although, an introduction to applied optics is not a pre-requisite for this course, it will be understood that most students are familiar with electromagnetic waves and optics related material covered in freshman/sophomore physics courses. Those who are not should seek the advice of the instructor immediately.

In this first lecture, we start by recapitulating the basic concepts of fields and waves which are essential to understand the electromagnetic theory of light. This theory is required to explain the two basic facts of optics not accounted for by the corpuscular or geometrical theory of light, namely: i) no matter how collimated, light beams always spread out after propagating a sufficient distance (diffraction of light), and ii) no matter how perfect a lens one uses, a parallel or 'perfectly' collimated beam of light can not be brought to a point focus.

1.2 Fields

By definition, a field is a collection of values for all of space and time. Mathematically, it is represented by a function which depends on both the spatial position denoted by $\bar{r}$ (overbar

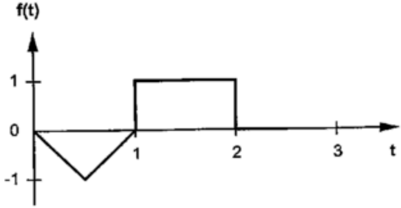

Figure 1.1: *A representative function $f(t)$ of time.*

indicates a vector quantity) and time denoted by t. A field can be static or dynamic, scalar or vector. A vector field requires a vector function of space and time for its description. The temperature field, used by meteorologists in weather prediction, is an example of a dynamic scalar field and the electric field inside the parallel plates of a capacitor is an example of a static vector field.

1.3 Waves

A wave is a field which propagates, i.e., the function describing the field gets displaced in a given direction (the wave propagation direction) with the passage of time. We illustrate this by way of an example. Consider a scalar field $u(\bar{r}, t)$, $\bar{r} = (x, y, z)$, which has the following functional form:

$$u(\bar{r}, t) = f(t - z/c) \tag{1.1}$$

with the function $f(t)$ as sketched in Fig. 1.1. Figure 1.2 sketches the field as a function of time both at $z = 0$ as well as $z = L$. We see that the field arrives at $z = L$ a time L/c later. Therefore, the function $f(t - z/c)$ describes a wave propagating in the $+z$ direction with speed c. This particular form represents a *uniform plane wave* because the function f has the same value for all values of x and y. In Fig. 1.3, we sketch the wave $f(t - z/c)$ as a function of z at $t = 0$. In the above example, the function f could represent pressure, electric field, or any other physical quantity exhibiting wave nature.

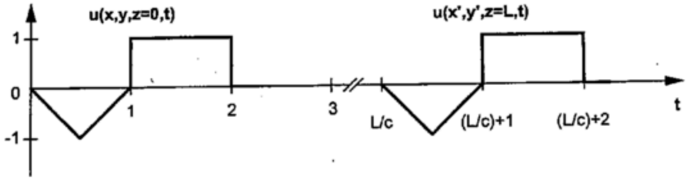

Figure 1.2: *The wave at $z = 0$ (left) and at $z = L$ (right).*

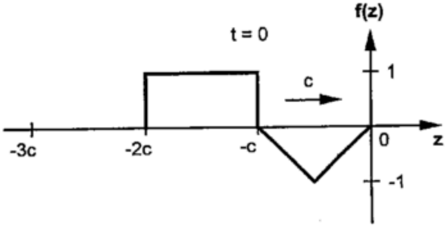

Figure 1.3: *The wave in space at $t = 0$.*

We now show that fields which are waves satisfy the following *wave equation*:

$$\nabla^2 u(\bar{r},t) - \frac{1}{c^2}\frac{\partial^2 u(\bar{r},t)}{\partial t^2} = 0, \tag{1.2}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.3}$$

denotes the Laplacian and is pronounced "del squared". Let $\xi = t - z/c$, then $u(\bar{r},t) = f(t - z/c) = f(\xi)$ implies that $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial t} = \frac{\partial f}{\partial \xi}$ and $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial z} = -\frac{1}{c}\frac{\partial f}{\partial \xi}$. Therefore, $\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial \xi^2}$ and $\frac{\partial^2 f}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial \xi^2}$. Also $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 0$. All these partial derivatives imply that $\nabla^2 u(\bar{r},t) = \frac{1}{c^2}\frac{\partial^2 u(\xi)}{\partial \xi^2}$ and $\frac{\partial^2 u(\bar{r},t)}{\partial t^2} = \frac{\partial^2 u(\xi)}{\partial \xi^2}$; clearly satisfying the wave equation (1.2).

The wave equation (1.2) has a very important property that it is *linear*, i.e., if $u_1(\bar{r},t)$ and $u_2(\bar{r},t)$ are two solutions of the wave equation, then $au_1(\bar{r},t) + bu_2(\bar{r},t)$ is also a solution where a and b are arbitrary constants. This property leads to the *principle of superposition*.

It can also be shown that $u(\bar{r},t) = f(t - \bar{r}\cdot\bar{i}_k/c)$ is a solution of the wave equation for any f, where $\bar{i}_k$ is an arbitrary unit vector. The wave $f(t - \bar{r}\cdot\bar{i}/c)$ propagates in the $\bar{i}$ direction with speed c and in the graphical example given above $\bar{i} = \bar{i}_z$.

In the next lecture, we will review a few properties of the *electromagnetic waves* which form the basis of the electromagnetic theory of light.

NORTHWESTERN UNIVERSITY
Department of Electrical and Computer Engineering

ECE 379 – Lecture 2

ELECTROMAGNETIC WAVES

Reading Assignment: YARIV – Secs. 1.1, 1.2, and 1.3.

2.1 Maxwell's Equations

We all know from the pioneering works of J. C. Maxwell, H. Hertz, and A. Einstein that light beams are electromagnetic waves and Maxwell's equations form the underlying basis for their description. In free space, devoid of all sources, Maxwell's equations take the following form in rationalized MKS units:

$$\nabla \times \bar{e}(\bar{r},t) = -\mu_0 \frac{\partial \bar{h}(\bar{r},t)}{\partial t}, \tag{2.1}$$

$$\nabla \times \bar{h}(\bar{r},t) = \epsilon_0 \frac{\partial \bar{e}(\bar{r},t)}{\partial t}, \tag{2.2}$$

$$\nabla \cdot \epsilon_0 \bar{e}(\bar{r},t) = 0, \tag{2.3}$$

$$\nabla \cdot \mu_0 \bar{h}(\bar{r},t) = 0. \tag{2.4}$$

Here $\bar{e}(\bar{r},t)$ and $\bar{h}(\bar{r},t)$ are the electric and the magnetic fields, respectively; $\nabla \equiv \bar{i}_x \frac{\partial}{\partial x} + \bar{i}_y \frac{\partial}{\partial y} + \bar{i}_z \frac{\partial}{\partial z}$ (pronounced as 'del') is the vector differential operator; $\cdot$ and $\times$ denote scalar and vector products, respectively, between two vector quantities; and ϵ_0 and μ_0, respectively, are the permittivity and permeability of free-space. In MKS units, $\bar{e}$ is in volts/meter (V/m), $\bar{h}$ is in amperes/meter (A/m), $4\pi\epsilon_0 = (9 \times 10^9)^{-1}$ farads/meter (F/m), and $\mu_0 = 4\pi \times 10^{-7}$ henrys/meter (H/m).

2.2 Wave Equation

Taking curl of Eq. (2.1) and using Eqs. (2.2) and (2.3), we get

$$\nabla \times \nabla \times \bar{e} = -\mu_0 \frac{\partial}{\partial t} \nabla \times \bar{h} = -\mu_0 \epsilon_0 \frac{\partial^2 \bar{e}}{\partial t^2}.$$

But $\nabla \times \nabla \times \bar{e} \equiv -\nabla^2 \bar{e} + \nabla(\nabla \cdot \bar{e}) = -\nabla^2 \bar{e}$, giving the following wave-equation for $\bar{e}(\bar{r}, t)$:

$$\nabla^2 \bar{e}(\bar{r}, t) - \frac{1}{c^2} \frac{\partial^2 \bar{e}(\bar{r}, t)}{\partial t^2} = 0, \tag{2.5}$$

where $c = \sqrt{1/\mu_0 \epsilon_0}$ is the speed of light in vacuum. In MKS units $c = 3 \times 10^8$ m/s. Similarly, one can obtain

$$\nabla^2 \bar{h}(\bar{r}, t) - \frac{1}{c^2} \frac{\partial^2 \bar{h}(\bar{r}, t)}{\partial t^2} = 0, \tag{2.6}$$

which is a wave equation for the magnetic field. Note that both Eqs. (2.5) and (2.6) are vector equations, meaning that each component of $\bar{e}$ and $\bar{h}$ satisfies the scalar wave equation (1.2). Also, $\bar{e}(\bar{r}, t)$ and $\bar{h}(\bar{r}, t)$ are coupled through the curl equations (2.1) and (2.2).

2.3 Properties of Electromagnetic Waves

Let us illustrate some of the properties of electromagnetic waves by way of a few examples. Consider the electric field associated with an electromagnetic wave to be $\bar{e}(\bar{r}, t) = f(t - \bar{r} \cdot \bar{i}_k/c)\bar{i}$, where $\bar{i}_k$ and $\bar{i}$ are two unit vectors, and $f(t - \bar{r} \cdot \bar{i}_k/c)$ represents a wave propagating in the $\bar{i}_k$ direction with speed c. From Eq. (2.3) we have

$$\nabla \cdot \epsilon_0 \bar{e}(\bar{r}, t) = \epsilon_0 \left[\bar{i}_x \cdot \bar{i} \frac{\partial f}{\partial x} + \bar{i}_y \cdot \bar{i} \frac{\partial f}{\partial y} + \bar{i}_z \cdot \bar{i} \frac{\partial f}{\partial z} \right] = 0.$$

Let $\xi = t - \bar{r} \cdot \bar{i}_k/c$ then $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial x} = -\frac{\alpha_x}{c} \frac{\partial f}{\partial \xi}$. Similarly $\frac{\partial f}{\partial y} = -\frac{\alpha_y}{c} \frac{\partial f}{\partial \xi}$ and $\frac{\partial f}{\partial z} = -\frac{\alpha_z}{c} \frac{\partial f}{\partial \xi}$ where α_x, α_y, and α_z are direction cosines of $\bar{i}_k$, i.e., $\bar{i}_k = \alpha_x \bar{i}_x + \alpha_y \bar{i}_y + \alpha_z \bar{i}_z$. Therefore, the above equation becomes

$$\nabla \cdot \epsilon_0 \bar{e}(\bar{r}, t) = -\frac{\epsilon_0}{c} \left[\alpha_x \bar{i}_x + \alpha_y \bar{i}_y + \alpha_z \bar{i}_z \right] \cdot \bar{i} \frac{\partial f}{\partial \xi} = 0,$$

which is true for any f implying that $\bar{i}_k \cdot \bar{i} = 0$, i.e., the electric field associated with an electromagnetic wave must be perpendicular to its direction of propagation. This is a very important result and we say that electromagnetic waves in a free space are *transverse waves*.

Let us consider another example for which

$$\bar{e}(\bar{r}, t) = [f(t - z/c) + g(t + z/c)]\bar{i}_x, \tag{2.7}$$

representing the superposition of a $+z$ going plane wave f and a $-z$ going plane wave g. Note that we have already assumed the transverse property. Then from Eq. (2.1), we have

$$-\mu_0 \frac{\partial \bar{h}(\bar{r},t)}{\partial t} = \nabla \times \bar{e}(\bar{r},t) = \begin{vmatrix} \bar{i}_x & \bar{i}_y & \bar{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e_x & e_y & e_z \end{vmatrix} = \bar{i}_y \frac{\partial e_x}{\partial z},$$

because $e_y = e_z = \frac{\partial e_x}{\partial y} = 0$. Therefore,

$$\frac{\partial \bar{h}(\bar{r},t)}{\partial t} = -\frac{1}{\mu_0}\left[-\frac{1}{c}\frac{\partial f}{\partial \xi} + \frac{1}{c}\frac{\partial g}{\partial \xi}\right]\bar{i}_y = \sqrt{\frac{\epsilon_0}{\mu_0}}\left[\frac{\partial f}{\partial \xi} - \frac{\partial g}{\partial \xi}\right]\bar{i}_y = \frac{\partial \bar{h}(\xi)}{\partial \xi},$$

which implies that

$$\bar{h}(\bar{r},t) = \sqrt{\frac{\epsilon_0}{\mu_0}}\left[f(t - z/c) - g(t + z/c)\right]\bar{i}_y, \tag{2.8}$$

i.e., the magnetic field associated with the electromagnetic wave is also a superposition of a $+z$ going wave and a $-z$ going wave with one important difference that $\bar{h}(\bar{r},t) \perp \bar{e}(\bar{r},t)$. Hence, not only the electromagnetic waves are transverse, but also the electric and magnetic fields associated with them are perpendicular to each other.

2.4 Poynting Vector

We all know that electromagnetic waves transport power which is best described by the Poynting vector $\bar{s}(\bar{r},t)$ defined by

$$\bar{s}(\bar{r},t) \equiv \bar{e}(\bar{r},t) \times \bar{h}(\bar{r},t), \tag{2.9}$$

giving the power density (W/m^2) carried by the wave at position $\bar{r}$ and time t in the direction of $\bar{s}$. The instantaneous power crossing a given surface σ is given by the surface integral $\int_\sigma \bar{s} \cdot \bar{i}_a da$, where $\bar{i}_a$ is a unit vector normal to the surface. For the second example considered in Sec. (2.3), Eqs. (2.7) and (2.8) give $\bar{s}(\bar{r},t) = \sqrt{\epsilon_0/\mu_0}\,[f^2(t - z/c) - g^2(t + z/c)]\bar{i}_z$. Thus, the power density flowing along the z direction is the difference of the power density carried by the $+z$ going wave and that carried by the $-z$ going wave.

2.5 Monochromatic Waves

In this course, we are interested in the light output of laser sources which is almost monochromatic. The electric field associated with the electromagnetic wave of such light can be written as

$$\bar{e}(\bar{r},t) = \text{Re}[\bar{E}(\bar{r})\exp(-j2\pi\nu t)] \tag{2.10}$$

where ν is the frequency of the monochromatic wave (3×10^{14} Hz at 1μm wavelength) and $\bar{E}(\bar{r})$ is the *complex* vector envelope at position $\bar{r}$. For example, the x-component of the electric-field is then given by

$$e_x(\bar{r},t) = |E_x(\bar{r})|\cos[2\pi\nu t - \phi_x(\bar{r})], \tag{2.11}$$

where we have introduced the polar coordinates for the complex quantity $E_x(\bar{r})$

$$E_x(\bar{r}) \equiv |E_x(\bar{r})|\exp[j\phi_x(\bar{r})]. \tag{2.12}$$

Note that ϕ_x is not the x-component of a vector $\bar{\phi}$. At any location $\bar{r}$, the field is cosinusoidally varying with frequency ν, justifying the monochromaticity condition.

After substituting Eq. (2.10) in the Ampere's Law, Eq. (2.2), we obtain $\nabla \times \bar{h}(\bar{r},t) = \epsilon_0 \frac{\partial}{\partial t}[\text{Re}\{\bar{E}(\bar{r})\exp(-j\omega t)\}] = \epsilon_0 \text{Re}[-j\omega \bar{E}(\bar{r})\exp(-j\omega t)]$, where $\omega = 2\pi\nu$, implying that $\bar{h}(\bar{r},t)$ must also have an exponential time dependence for a monochromatic electromagnetic wave allowing us to write

$$\bar{h}(\bar{r},t) = \text{Re}[\bar{H}(\bar{r})\exp(-j2\pi\nu t)]. \tag{2.13}$$

For monochromatic waves, the Faraday's Law and the Ampere's Law equations (2.1) and (2.2) then simplify to

$$\nabla \times \bar{E}(\bar{r}) = j\mu_0\omega\bar{H}(\bar{r}), \tag{2.14}$$

$$\nabla \times \bar{H}(\bar{r}) = -j\epsilon_0\omega\bar{E}(\bar{r}) \tag{2.15}$$

and the wave equations (2.5) and (2.6) reduce to the following *Helmholtz* equations:

$$\nabla^2\bar{E}(\bar{r}) + \frac{\omega^2}{c^2}\bar{E}(\bar{r}) = 0, \tag{2.16}$$

$$\nabla^2 \bar{H}(\bar{r}) \ + \ \frac{\omega^2}{c^2}\bar{H}(\bar{r}) = 0. \tag{2.17}$$

Also from Eqs. (2.3) and (2.4), $\nabla \cdot \bar{E}(\bar{r}) = 0$ and $\nabla \cdot \bar{H}(\bar{r}) = 0$.

2.6 Intensity of Monochromatic Waves (Irradiance)

Optical detectors, including photocells, human eyes, and photographic plates, respond to optical intensity which is defined as the magnitude of the time average of the Poynting vector. Using Eqs. (2.9), (2.10), and (2.13), the intensity of a monochromatic wave at point $\bar{r}$ is given by

$$\begin{aligned} I(\bar{r}) &= |\langle \bar{s}(\bar{r},t)\rangle| \\ &= \frac{1}{T}\left|\int_{-T/2}^{T/2} dt\, \mathrm{Re}[\bar{E}(\bar{r})\exp(-j\omega t)] \times \mathrm{Re}[\bar{H}(\bar{r})\exp(-j\omega t)]\right|, \end{aligned} \tag{2.18}$$

where $T = 2\pi/\omega$ is the time period of an optical cycle. Evaluating the time integral, we get

$$I(\bar{r}) = \frac{1}{2}\left|\mathrm{Re}[\bar{E}(\bar{r}) \times \bar{H}^*(\bar{r})]\right| \equiv \left|\mathrm{Re}[\bar{S}(\bar{r})]\right|, \tag{2.19}$$

where $\bar{S}(\bar{r})$ is the complex Poynting vector, and the magnitude is to be understood as that of the vector quantity inside the square brackets.

2.7 Monochromatic Plane Waves

For such waves $\bar{E}(\bar{r})$ of Eq. (2.10) can be taken as $\bar{E}_0 \exp(j\bar{k}\cdot\bar{r})$, where $\bar{E}_0$ is a constant vector independent of position coordinate $\bar{r}$. Equation (2.10) can then be written as

$$\bar{e}(\bar{r},t) = \ \mathrm{Re}[\bar{E}_0 \exp\{-j(\omega t - \bar{k}\cdot\bar{r})\}], \tag{2.20}$$

which upon introducing real and imaginary parts of the complex vector $\bar{E}_0 \equiv \bar{E}_r + j\bar{E}_i$ becomes

$$\bar{e}(\bar{r},t) = \bar{E}_r \cos[2\pi\nu(t - \frac{\bar{i}_k \cdot \bar{r}}{(2\pi\nu/k)})] + \bar{E}_i \sin[2\pi\nu(t - \frac{\bar{i}_k \cdot \bar{r}}{(2\pi\nu/k)})], \tag{2.21}$$

where the wavevector $\bar{k} = k\bar{i}_k$ with $\bar{i}_k$ a unit vector along $\bar{k}$. The functional form of the above equation is $f(t - \bar{i}_k \cdot \bar{r}/c)$ with $c = 2\pi\nu/k$, identifying it to be a uniform plane wave propagating along the $\bar{i}_k$ direction with speed c. Since $\bar{e}(\bar{r},t)$ of Eq. (2.20) must also satisfy the wave equation (2.5), the constant c introduced above must be the speed of light in free space. The relation $c = 2\pi\nu/k$ then also defines $2\pi/k$ to be the wavelength λ of light.

2.8 Polarization Properties of Monochromatic Plane Waves

The vectors $\bar{E}_r$ and $\bar{E}_i$ give the polarization properties of the plane wave (2.20). In general, the polarization is elliptical. However, when $|\bar{E}_r \times \bar{E}_i| = 0$, then the polarization is linear; and when $\bar{E}_r \cdot \bar{E}_i = 0$ with $|\bar{E}_r| = |\bar{E}_i|$, then the polarization is circular. Furthermore, in the latter case if $(\bar{E}_r \times \bar{E}_i) \cdot \bar{i}_k > 0$, it is right circular and if $\bar{E}_r \times \bar{E}_i \cdot \bar{i}_k < 0$, it is left circular.

For monochromatic plane waves Eqs. (2.14) and (2.15) further simplify to

$$\bar{k} \times \bar{E}_0 = \mu_0 \omega \bar{H}_0, \tag{2.22}$$

$$\bar{k} \times \bar{H}_0 = -\epsilon_0 \omega \bar{E}_0, \tag{2.23}$$

with the spatial dependence for the magnetic field in Eq. (2.13) being given by $\bar{H}(\bar{r}) = \bar{H}_0 \exp(j\bar{k}\cdot\bar{r})$. Also, for monochromatic plane waves in free space, Eq. (2.19) together with Eq. (2.20) leads to the following simple expression for its intensity:

$$I = \frac{1}{2}\epsilon_0 c |\bar{E}_0|^2. \tag{2.24}$$

We finish this lecture by re-writing the plane wave electric field in a slightly different form:

$$\begin{aligned}
\bar{e}(\bar{r},t) &= \mathrm{Re}[\bar{E}_0 \exp\{-j(\omega t - \bar{k}\cdot\bar{r})\}] \\
&= \bar{i}_x |E_{0x}| \cos(\omega t - \bar{k}\cdot\bar{r} - \phi_{0x}) \\
&+ \bar{i}_y |E_{0y}| \cos(\omega t - \bar{k}\cdot\bar{r} - \phi_{0y}) \\
&+ \bar{i}_z |E_{0z}| \cos(\omega t - \bar{k}\cdot\bar{r} - \phi_{0z}),
\end{aligned} \tag{2.25}$$

where $\bar{E}_0 = E_{0x}\bar{i}_x + E_{0y}\bar{i}_y + E_{0z}\bar{i}_z = \bar{i}_x|E_{0x}|\exp(j\phi_{0x}) + \bar{i}_y|E_{0y}|\exp(j\phi_{0y}) + \bar{i}_z|E_{0z}|\exp(j\phi_{0z})$. In the remainder of this course, we will assume all electromagnetic waves to be linearly polarized along $\bar{i}_y$ unless otherwise noted so that $E_{0x} = E_{0z} = 0$. The wavevector $\bar{k}$ must then lie in the x-z plane and will generally be assumed parallel to the z direction. In the next lecture, we will consider the reflection and refraction of an electromagnetic wave at a dieletric interface.

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NORTHWESTERN UNIVERSITY

Department of Electrical Engineering and Computer Science

EECS 379 – Lecture 3

REFLECTION AND REFRACTION

AT A DIELECTRIC INTERFACE

3.1 Electromagnetic Waves in Dielectric Media

In a source-free nonmagnetic and transparent medium with dielectric constant $\epsilon = \epsilon_0 n^2$, where n is the refractive index of the medium, the Maxwell's equations (2.1)-(2.4) with ϵ_0 replaced by ϵ lead to the following wave equation:

$$\nabla^2 \bar{e}(\bar{r}, t) - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\bar{e}(\bar{r}, t) = 0, \tag{3.1}$$

describing the propagation of electromagnetic waves with speed $v = c/n$ in that medium. As an electromagnetic wave enters the dielectric medium, the boundary condition at the dielectric interface dictates that the frequency ν of the wave must remain unchanged. Later in this lecture, we will clarify this point further. Therefore, the wavelength of the wave in the medium becomes $v/\nu = (c/n)/\nu = \lambda/n$, where λ is the free-space wavelength defined in the previous lecture. The electric field associated with a monochromatic plane wave in the medium can then be written as

$$\bar{e}(\bar{r}, t) = \mathrm{Re}[\bar{E}_0 \exp\{-j(\omega t - \bar{K} \cdot \bar{r})\}], \tag{3.2}$$

which is the same as that given by Eq. (2.20) with one difference that the magnitude of the wavevector $\bar{K}$ is now given by $|\bar{K}| = K = 2\pi/(\lambda/n) = 2\pi n/\lambda = nk$ (recall that in free space $k = 2\pi/\lambda$). In a uniform dielectric medium, all other properties of electromagnetic waves discussed in the previous lecture remain the same and Eqs. (2.22) and (2.23) become

$$\begin{aligned} \bar{K} \times \bar{E}_0 &= \mu_0 \omega \bar{H}_0, & (3.3) \\ \bar{K} \times \bar{H}_0 &= -n^2 \epsilon_0 \omega \bar{E}_0. & (3.4) \end{aligned}$$

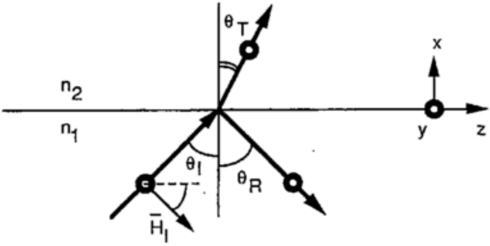

Figure 3.1: *Reflection and refraction at a dielectric interface.*

The intensity expression (2.24) modifies to

$$I = \frac{1}{2}\epsilon_0 v n^2 |\bar{E}_0|^2. \tag{3.5}$$

3.2 Fresnel Problem

We know from practical experience that when a beam of light hits a dielectric interface, such as a piece of glass, a part of the beam is reflected back and a part is transmitted obeying Snell's laws. In this section, we derive these laws together with the fractions of the beam that are reflected and transmitted. Consider a plane wave passing through a dielectric interface as shown in Fig. 3.1. The arrows indicate directions of propagation of the incident plane wave at an angle θ_I (relative to the interface-normal), the reflected wave at θ_R and the transmitted wave at θ_T. The electric and magnetic fields associated with the incident plane wave are then given by

$$\bar{e}_I(\bar{r},t) = \mathrm{Re}[\bar{E}_I \exp\{-j(\omega t - \bar{K}_I \cdot \bar{r})\}], \tag{3.6}$$

$$\bar{h}_I(\bar{r},t) = \mathrm{Re}[\bar{H}_I \exp\{-j(\omega t - \bar{K}_I \cdot \bar{r})\}], \tag{3.7}$$

where $\bar{H}_I$ can be found using Eq. (3.3), i.e.,

$$\bar{H}_I = \frac{1}{\mu_0 \omega}\bar{K}_I \times \bar{E}_I, \tag{3.8}$$

and $\bar{K}_I$ can be decomposed into x and z components as

$$\bar{K}_I = K_I(\bar{i}_x \cos \theta_I + \bar{i}_z \sin \theta_I), \tag{3.9}$$

with $K_I = 2\pi n_1/\lambda$. Similarly for the reflected and transmitted fields, we have

$$\begin{aligned}
\bar{e}_R(\bar{r},t) &= \mathrm{Re}[\bar{E}_R \exp\{-j(\omega t - \bar{K}_R \cdot \bar{r})\}], && (3.10)\\
\bar{h}_R(\bar{r},t) &= \mathrm{Re}[\bar{H}_R \exp\{-j(\omega t - \bar{K}_R \cdot \bar{r})\}], && (3.11)\\
\bar{H}_R &= \frac{1}{\mu_0\omega}\bar{K}_R \times \bar{E}_R, && (3.12)\\
\bar{K}_R &= K_R(-\bar{i}_x \cos\theta_R + \bar{i}_z \sin\theta_R), && (3.13)\\
\bar{e}_T(\bar{r},t) &= \mathrm{Re}[\bar{E}_T \exp\{-j(\omega t - \bar{K}_T \cdot \bar{r})\}], && (3.14)\\
\bar{h}_T(\bar{r},t) &= \mathrm{Re}[\bar{H}_T \exp\{-j(\omega t - \bar{K}_T \cdot \bar{r})\}], && (3.15)\\
\bar{H}_T &= \frac{1}{\mu_0\omega}\bar{K}_T \times \bar{E}_T, && (3.16)\\
\bar{K}_T &= K_T(\bar{i}_x \cos\theta_T + \bar{i}_z \sin\theta_T), && (3.17)
\end{aligned}$$

with $K_R = 2\pi n_1/\lambda$ and $K_T = 2\pi n_2/\lambda$. We further assume the transverse electric (TE) case, i.e., the electric-field vector of the incident wave is perpendicular to the plane of incidence. Therefore,

$$\bar{E}_I = E_I\,\bar{i}_y. \tag{3.18}$$

Now we consider the consequences of the boundary conditions (at the $x = 0$ interface) that the tangential components of $\bar{E}$ and $\bar{H}$ be continuous. In fact, as pointed out earlier, we have already used these boundary conditions (the fact that they be satisfied at all times) to write the monochromatic time dependence, with angular frequency ω, of $\bar{e}_R(\bar{r},t)$, $\bar{h}_R(\bar{r},t)$, $\bar{e}_T(\bar{r},t)$ and $\bar{h}_T(\bar{r},t)$ in Eqs. (3.10), (3.11), (3.14), and (3.15) respectively. Furthermore, these boundary conditions must be satisfied at all points in the $x = 0$ plane. This can happen only when the spatial dependence of both the reflected and the transmitted fields is identical to that of the incident field at $x = 0$. The fact that they be satisfied at all values of y was used in writing Eqs. (3.13) and (3.17) where we assumed that both $\bar{K}_R$ and $\bar{K}_T$ lie in the plane of incidence – the plane containing the incident wave and the normal to the interface – which in our case is the x-z plane. Also, using Eqs. (3.6), (3.9), (3.10), and (3.13), we must have $\exp(j\frac{2\pi}{\lambda}n_1 z\sin\theta_I) = \exp(j\frac{2\pi}{\lambda}n_1 z\sin\theta_R)$, giving the *law of reflection*

$$\theta_I = \theta_R, \tag{3.19}$$

i.e., the angle of incidence is equal to the angle of reflection. Similarly, using Eqs. (3.6), (3.9), (3.14), and (3.17), we must have $\exp(j\frac{2\pi}{\lambda}n_1 z\sin\theta_I) = \exp(j\frac{2\pi}{\lambda}n_2 z\sin\theta_T)$, giving the *Snell's law of refraction*

$$n_1\sin\theta_I = n_2\sin\theta_T. \tag{3.20}$$

Using the above derived Snell's laws and Eqs. (3.6), (3.10), and (3.14), in order for the tangential component of the electric field to be continuous on either side of the $x = 0$ plane, we must have

$$\begin{aligned} E_I + \bar{E}_R\cdot\bar{i}_y &= \bar{E}_T\cdot\bar{i}_y, & (3.21)\\ 0 + \bar{E}_R\cdot\bar{i}_z &= \bar{E}_T\cdot\bar{i}_z & (3.22)\end{aligned}$$

Similarly, using Eqs. (3.7), (3.11), and (3.15), the continuity condition for the tangential components of the magnetic field implies

$$\begin{aligned} (\bar{H}_R + \bar{H}_I)\cdot\bar{i}_y &= \bar{H}_T\cdot\bar{i}_y, & (3.23)\\ (\bar{H}_R + \bar{H}_I)\cdot\bar{i}_z &= \bar{H}_T\cdot\bar{i}_z. & (3.24)\end{aligned}$$

Furthermore, using Eqs. (3.8), (3.12), and (3.16) these become

$$\begin{aligned} n_1[(\bar{i}_R\times\bar{E}_R)\cdot\bar{i}_y + (\bar{i}_I\times\bar{E}_I)\cdot\bar{i}_y] &= n_2(\bar{i}_T\times\bar{E}_T)\cdot\bar{i}_y, & (3.25)\\ n_1[(\bar{i}_R\times\bar{E}_R)\cdot\bar{i}_z + (\bar{i}_I\times\bar{E}_I)\cdot\bar{i}_z] &= n_2(\bar{i}_T\times\bar{E}_T)\cdot\bar{i}_z. & (3.26)\end{aligned}$$

Equations (3.22) and (3.23) or (3.25) can both be satisfied only if $\bar{E}_R = E_R\,\bar{i}_y$ and $\bar{E}_T = E_T\,\bar{i}_y$. This is because in Eq. (3.23) $\bar{H}_I\cdot\bar{i}_y = 0$. Equations (3.21) and (3.26) then give the following linear equations in E_R and E_T

$$\begin{aligned} E_I + E_R &= E_T, & (3.27)\\ n_1(E_I\cos\theta_I - E_R\cos\theta_R) &= n_2 E_T\cos\theta_T, & (3.28)\end{aligned}$$

which can be solved using Eqs. (3.19) and (3.20) to give the following expressions for the *field* reflection and transmission coefficients:

$$r \equiv \frac{E_R}{E_I} = \frac{n_1 \cos\theta_I - \sqrt{n_2^2 - (n_1 \sin\theta_I)^2}}{n_1 \cos\theta_I + \sqrt{n_2^2 - (n_1 \sin\theta_I)^2}}, \tag{3.29}$$

$$t \equiv \frac{E_T}{E_I} = \frac{2n_1 \cos\theta_I}{n_1 \cos\theta_I + \sqrt{n_2^2 - (n_1 \sin\theta_I)^2}}, \tag{3.30}$$

respectively. Note that both r and t can be complex. Using Eqs. (3.29) and (3.5), we define the following *intensity* reflection coefficient:

$$R \equiv \left|\frac{E_R}{E_I}\right|^2 = |r|^2. \tag{3.31}$$

3.3 Evanescent Waves

When θ_I is greater than the critical angle defined by $\theta_C = \sin^{-1}(n_2/n_1)$ for $n_2 < n_1$, we know from geometrical optics that there is no transmitted wave. However, the electric and magnetic fields do not vanish in the $x > 0$ region. Using Eqs. (3.14), (3.17), (3.20), and (3.30), for $\theta_I > \theta_C$, the electric field is given by

$$\bar{E}_T(r) = \mathrm{Re}\left[t\ \bar{E}_I \exp\left\{-\frac{2\pi n_2 x}{\lambda}\sqrt{\frac{n_1^2 \sin^2\theta_I}{n_2^2} - 1}\right\} \exp\left\{\frac{j2\pi n_1 z \sin\theta_I}{\lambda}\right\}\right] \tag{3.32}$$

and the magnetic field can be obtained using Eqs. (3.15) and (3.16). We see that the field decays exponentially along x (for $x > 0$) with a decay constant $(2\pi/\lambda)\sqrt{(n_1 \sin\theta_I)^2 - n_2^2}$ and propagates along the interface (even in the $x > 0$ region). The field in the $x > 0$ region is called the *evanescent field*.

In the next lecture, we will use the field reflection and transmission coefficients derived in this lecture to analyze a Fabry-Perot etalon.

NORTHWESTERN UNIVERSITY

Department of Electrical Engineering and Computer Science

Lecture 4 – EECS 379

OPTICAL RESONATORS

Reading Assignment: YARIV – Secs. 2.1, 2.2, 4.0, 4.1, and 4.2.

4.1 Fabry-Perot Etalon

A Fabry-Perot (FP) etalon is a slab of glass or any other transparent material whose two surfaces are optically flat (i.e., the extent of any surface roughness, scratch, or pit is much smaller than a wavelength of light) and parallel to each other. Due to multiple reflections at the two surfaces and interference between the multiply reflected waves, such an etalon shows interesting reflection and transmission behavior as a function of wavelength of the incident light beam. In this section, we study this phenomenon for the transverse electric (TE) case.

Consider a monochromatic plane wave incident on an FP etalon, of thickness ℓ and refractive index n, placed at $z = 0$ making an angle θ_I with the z axis as shown in Fig. 4.1.

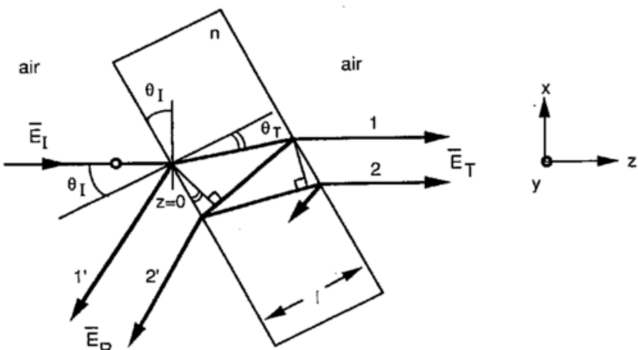

Figure 4.1: *A Fabry-Perot Etalon.*

The incident field can be written as

$$\bar{e}_I(\bar{r},t) = \mathrm{Re}[\bar{E}_I \exp\{-j(\omega t - \bar{k}_I \cdot \bar{r})\}], \tag{4.1}$$

where $\bar{E}_I = E_I \bar{i}_y$ and $\bar{k}_I = (2\pi/\lambda)\bar{i}_z$. Using Eqs. (3.29) and (3.30), the field reflection and transmission coefficients for the air-glass interface are given by $(n_1 = 1, n_2 = n)$

$$r_1 = \frac{\cos\theta_I - \sqrt{n^2 - \sin^2\theta_I}}{\cos\theta_I + \sqrt{n^2 - \sin^2\theta_I}}, \tag{4.2}$$

$$t_1 = \frac{2\cos\theta_I}{\cos\theta_I + \sqrt{n^2 - \sin^2\theta_I}}. \tag{4.3}$$

Similarly, for the glass-air interface, $n_1 = n$ and $n_2 = 1$ give

$$r_2 = \frac{n\cos\theta_T - \sqrt{1 - n^2\sin^2\theta_T}}{n\cos\theta_T + \sqrt{1 - n^2\sin^2\theta_T}}, \tag{4.4}$$

(note that in this case the angle of incidence is θ_T)

$$t_2 = \frac{2n\cos\theta_T}{n\cos\theta_T + \sqrt{1 - n^2\sin^2\theta_T}}. \tag{4.5}$$

As explained in the previous lecture, both the reflected and transmitted fields must be y polarized and have the same frequency and wavelength (they both are in air). Their electric fields can then be written as

$$\bar{e}_R(\bar{r},t) = \mathrm{Re}[\bar{E}_R \exp\{-j(\omega t - \bar{k}_R \cdot \bar{r})\}] \tag{4.6}$$

for $z \le 0$ and

$$\bar{e}_T(\bar{r},t) = \mathrm{Re}[\bar{E}_T \exp\{-j(\omega t - \bar{k}_T \cdot \{\bar{r} - \quad \bar{\mathrm{r}}|_{\mathrm{AB}} \qquad \})\}] \tag{4.7}$$

for $z \ge \ell/\cos\theta_T$. Here $\bar{k}_R = (2\pi/\lambda)[-\bar{i}_x \sin 2\theta_I - \bar{i}_z \cos 2\theta_I]$, $\bar{k}_T = (2\pi/\lambda)\bar{i}_z$ (because the transmitted rays are parallel to the incident rays), $\bar{E}_R = E_R \bar{i}_y$, and $\bar{E}_T = E_T \bar{i}_y$. The complex envelopes E_R and E_T of the reflected and transmitted waves, respectively, consist of superpositions of the multiply reflected and transmitted waves and can be obtained from E_I using Eqs. (4.2) to (4.5). For example, for the first and second transmitted waves 1 and 2, respectively (cf. Fig 4.1), $E_T^1 = E_I t_1 t_2 \exp(jnk\ell/\cos\theta_T)$ and $E_T^2 = E_T^1 (r_2/t_2) r_2 t_2 \exp(j\delta) =$

$E_T^1 r_2^2 \exp(j\delta)$, where $\delta = (2\pi/\lambda)[(2\ell n/\cos\theta_T) - 2\ell\tan\theta_T(n\sin\theta_T)] = k2n\ell\cos\theta_T$. Similarly, for the first and second reflected waves 1′ and 2′, respectively, $E_R^1 = r_1 E_I$ and $E_R^2 = (E_R^1/r_1)\, t_1 r_2 t_2 \exp(j\delta)$. Summing the infinite number of reflected and transmitted waves, we get

$$\begin{aligned} E_R &= E_I[r_1 + r_2 t_1 t_2 \exp(j\delta) + r_2 t_1 t_2 r_2^2 \exp(2j\delta) + \ldots] \\ &= E_I\left[r_1 + \frac{r_2 t_1 t_2 \exp(j\delta)}{1 - r_2^2\exp(j\delta)}\right], \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} E_T &= E_I t_1 t_2 \exp(jnk\ell/\cos\theta_T)[1 + r_2^2\exp(j\delta) + r_2^4\exp(2j\delta) + \ldots] \\ &= E_I t_1 t_2 \frac{\exp(jnk\ell/\cos\theta_T)}{1 - r_2^2\exp(j\delta)}. \end{aligned} \tag{4.9}$$

Using Snell's law in Eqs. (4.2)-(4.5), it is easily seen that $r_1 = -r_2$ and $-r_1 r_2 + t_1 t_2 = 1$; the latter being a statement of the conservation of energy. Equations (4.8) and (4.9) then further simplify, giving the total field reflection and transmission coefficients

$$\begin{aligned} r_T \equiv \frac{E_R}{E_I} &= r_1 - \frac{r_1(1 - r_1^2)\exp(j\delta)}{1 - r_1^2\exp(j\delta)} \\ &= \frac{r_1[1 - \exp(j\delta)]}{1 - r_1^2\exp(j\delta)}, \end{aligned} \tag{4.10}$$

and

$$t_T \equiv \frac{E_T}{E_I} = \frac{\exp(jnk\ell/\cos\theta_T)[1 - r_1^2]}{1 - r_1^2\exp(j\delta)}. \tag{4.11}$$

Similar to Eq. (3.31), the total intensity reflection and transmission coefficients become

$$|r_T|^2 \equiv R_T = \frac{4R\sin^2(\delta/2)}{(1-R)^2 + 4R\sin^2(\delta/2)}, \tag{4.12}$$

$$|t_T|^2 \equiv T_T = \frac{(1-R)^2}{(1-R)^2 + 4R\sin^2(\delta/2)}, \tag{4.13}$$

where $\delta = \frac{4\pi\nu}{c} n\ell\cos\theta_T$, and $R = r_1^2$ is assumed real, which from Eq. (4.2) will always be the case because $n \geq 1$. It is easily verified that $R_T + T_T = 1$. That is, for a transparent etalon, whatever intensity is not reflected must be transmitted.

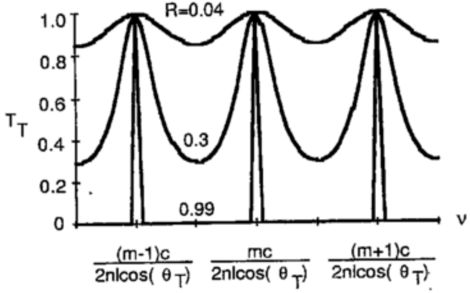

Figure 4.2: *Spectral behavior of the transmission coefficient.*

4.2 Spectral Behavior of the Transmission Coefficient

T_T of Eq. (4.13) is a periodic function in δ. Either ν, n, ℓ, or θ_T can be changed to vary δ. For the purpose of illustration, Fig. 4.2 shows T_T as a function of ν. T_T achieves a peak value of unity whenever $\delta/2 = m\pi$ or $(2\pi/\lambda)n\ell\cos\theta_T = m\pi$ for m an integer, or

$$n\ell\cos\theta_T = m\frac{\lambda}{2} \qquad \text{or} \qquad \nu = \frac{mc}{2n\ell\cos\theta_T}. \tag{4.14}$$

The spectral separation between two successive peaks is called the *free spectral range* (FSR) and is given by

$$\mathrm{FSR} = \frac{(m+1)c}{2n\ell\cos\theta_T} - \frac{mc}{2n\ell\cos\theta_T} = \frac{c}{2n\ell\cos\theta_T}. \tag{4.15}$$

Similarly, whenever Eq. (4.14) is satisfied, R_T goes to 0. That is, for discrete wavelengths, it appears as if the etalon did not exist as far as the incident light is concerned. The peaks in transmission are referred to as resonances of the etalon. The full width at half maximum (FWHM) $\Delta\nu_{1/2}$ of each peak is given by

$$\begin{aligned} \Delta\nu_{1/2} &= \sin^{-1}\left[\frac{(1-R)}{2\sqrt{R}}\right]\frac{c}{\pi n\ell\cos\theta_T} \\ &\overset{R\simeq 1}{\simeq} \frac{(1-R)c}{2\pi n\ell\sqrt{R}\cos\theta_T} \end{aligned} \tag{4.16}$$

and defines the finesse F of the etalon, which using Eqs. (4.14) and (4.15) is

$$F \equiv \frac{\mathrm{FSR}}{\Delta\nu_{1/2}} = \frac{\pi}{2\sin^{-1}[\frac{1-R}{2\sqrt{R}}]}$$
$$\stackrel{R\simeq 1}{\approx} \frac{\pi\sqrt{R}}{1-R}. \tag{4.17}$$

For a glass etalon at normal incidence, $R = 0.04$; but with thin film coatings, R can be increased to 0.99 or better giving an F of over 300 for such an etalon.

4.3 Fabry-Perot Etalons as Optical Filters

The fact that the transmission of an FP etalon is frequency or wavelength dependent allows them to be used as optical filters. If, for example, the incident light consisted of two monochromatic waves at frequencies ν_1 and ν_2, then by choosing n, ℓ, or θ_T such that $\nu_1 = \dfrac{mc}{2n\ell\cos\theta_T}$ for some integer m but $\dfrac{2n\ell\cos\theta_T}{c}\nu_2$ is not equal to an integer, the light with frequency ν_2 can be reflected away whereas that with ν_1 can be transmitted through the etalon unattenuated.

4.4 Optical Spectrum Analyzer

A high finesse FP etalon is often used as an optical spectrum analyzer. If the incident light is polychromatic and band-limited to less than one FSR, then only that frequency is transmitted for which the resonance condition (4.14) is satisifed. By monitoring the intensity of the transmitted light and varying the length (n or θ_T could be varied also) of the etalon, the spectrum of the incident light is easily measured. We will demonstrate this property in the laboratory.

4.5 Fabry-Perot Cavity

An FP cavity or a resonator can be formed with two plane mirrors (we will generalize to curved mirrors in the later part of the course) of reflectivity R placed parallel to each other as shown in Fig. 4.3. If the mirrors are lossless, then the transmissivity $T = 1 - R$. The system of Fig. 4.3 behaves like an FP etalon with $n = 1$ and $\theta_T = 0$. Equations (4.12)

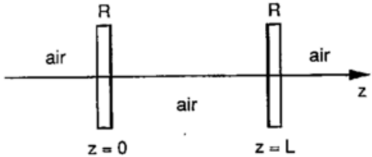

Figure 4.3: *A plane mirror Fabry-Perot cavity.*

through (4.17) then apply and the FSR = $c/2\ell$. For example, if $\ell = 15$cm (a typical value used in many helium-neon lasers), then FSR $= \dfrac{3 \times 10^8}{2 \times 0.15} = 1$GHz. Because of the multiple reflections between the two mirrors, a FP cavity can be thought of as a feedback providing device to anything placed inside the two mirrors and interacting with the optical field.

In the next lecture we will study the interaction of light with atomic sytems in order to understand the gain mechanism of a laser.