NORTHWESTERN UNIVERSITY

Department of Electrical Engineering and Computer Science

Lecture 5 – EECS 379

INTERACTION OF LIGHT WITH MATTER

Reading Assignment: YARIV – Sec. 5.4.

5.1 Introduction

A laser is nothing but an optical frequency oscillator. We all know that to make an oscillator, one needs an amplifier to provide gain and a positive feedback mechanism. As pointed out in the previous lecture, the feedback is provided with a Fabry-Perot resonator. The gain is obtained via interaction of light with a material medium resulting in Light Amplification by Stimulated Emission of Radiation (LASER).

5.2 Light Absorption and Emission

The wave equation (3.1) applies in a transparent dielectric medium, a medium which does not absorb or emit light. Equation (3.1) is easily generalized to an absorbing or emitting medium by using the constitutive relation

$$\bar{d}(\bar{r},t) = \epsilon_0 \bar{e}(\bar{r},t) + \bar{p}(\bar{r},t) \tag{5.1}$$

between the electric displacement vector $\bar{d}(\bar{r},t)$, the applied electric field $\bar{e}(\bar{r},t)$, and the induced polarization (induced dipole moment per unit volume) $\bar{p}(\bar{r},t)$ in the medium. Using (5.1) and repeating the calculation of Sec. 2.2, the following inhomogeneous wave equation is obtained

$$\nabla^2 \bar{e}(\bar{r},t) - \frac{1}{c^2}\frac{\partial^2 \bar{e}(\bar{r},t)}{\partial t^2} = \mu_0 \frac{\partial^2 \bar{p}(\bar{r},t)}{\partial t^2}. \tag{5.2}$$

For a monochromatic applied field, the induced polarization is also monochromatic and similar to Eq. (2.10), $\bar{p}(\bar{r},t)$ can be written as

$$\bar{p}(\bar{r},t) = \mathrm{Re}[\bar{P}(\bar{r})\exp(-j2\pi\nu t)]. \tag{5.3}$$

Furthermore, in an isotropic medium at any point $\bar{r}$, the induced polarization, to the lowest order, is proportional to the applied field. Thus we can write

$$\bar{P}(\bar{r}) = \epsilon_0 \chi \bar{E}(\bar{r}), \tag{5.4}$$

defining the electric susceptibility χ for the medium. In general χ is complex and depends upon the frequency ν of the applied field, i.e., $\chi = \chi(\nu)$. Substituting (2.10), (5.3), and (5.4) in Eq. (5.2), we obtain

$$\begin{aligned} \nabla^2 \bar{E}(\bar{r}) + \frac{\omega^2}{c^2}\bar{E}(\bar{r}) &= -\mu_0 \omega^2 \bar{P}(\bar{r}) \\ &= -\frac{\omega^2}{c^2}\chi(\nu)\bar{E}(\bar{r}), \end{aligned} \tag{5.5}$$

which is a generalization of the Helmholtz equation (2.16).

We now show that a $+z$ propagating plane wave

$$\bar{E}(\bar{r}) = \bar{i}_x E_0 \exp[(\alpha + j\beta)z] \tag{5.6}$$

whose envelope grows or decays (depending upon the sign of α) is a solution of (5.5). Direct substitution shows that (5.6) is a solution of (5.5) if $(\alpha + j\beta)^2 + \frac{\omega^2}{c^2} = -\frac{\omega^2}{c^2}[\chi'(\nu) + j\chi''(\nu)]$, where we have defined real and imaginary parts of $\chi(\nu)$ via

$$\chi(\nu) \equiv \chi'(\nu) + j\chi''(\nu). \tag{5.7}$$

Comparing the real and imaginary parts, we must then have

$$\alpha^2 - \beta^2 + k^2(1 + \chi'(\nu)) = 0, \tag{5.8}$$

$$\alpha = -\frac{k^2}{2\beta}\chi''(\nu). \tag{5.9}$$

Substituting (5.9) in (5.8), we further obtain

$$\beta^2 = \frac{k^2}{2}\left[\{1 + \chi'(\nu)\} \pm \sqrt{\{1 + \chi'(\nu)\}^2 + \{\chi''(\nu)\}^2}\right], \tag{5.10}$$

which for $\chi''(\nu) \ll 1$ simplifies to (dropping $-$ sign because $\beta^2 > 0$)

$$\beta^2 \simeq k^2[1 + \chi'(\nu)] \equiv n^2 k^2 = K^2. \tag{5.11}$$

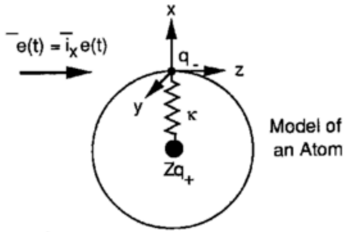

Figure 5.1: *Harmonic oscillator model of an atom.*

Thus, in a medium characterized by the complex susceptibility $\chi(\nu)$, a plane wave of frequency ν, given by (5.6), will propagate along the $+z$ direction seeing a refractive index $\sqrt{1+\chi'(\nu)}$ and its envelope will amplify if $\chi''(\nu) < 0$, or will decay if $\chi''(\nu) > 0$, or will stay unchanged if $\chi''(\nu) = 0$ which being the case for transparent media. Next we show that in ordinary media $\chi''(\nu) > 0$, *always.*

5.3 Electron-Oscillator Model of a Medium

Every medium consists of atoms which further consist of heavy nuclii and light electrons orbiting the former. When light enters the medium, under the action of its electric field oscillating at frequency ν the electrons also start to oscillate at frequency ν relative to the nuclii which stay at rest because of their large mass. The oscillating electrons re-radiate light damping themselves in the process. When light leaves the medium, the electrons, which are bound to the atoms, continue to oscillate at their natural frequencies eventually coming to rest due to radiative damping. Thus the interaction of light with matter can be modelled simply by the action of the light's electric field on a collection of electrons bound to springs of spring-constant κ as shown in Fig. 5.1. The equation of motion for the position x of the electron can then be written as (because $\bar{e} \propto \bar{i}_x$, there is no y or z motion)

$$m\frac{d^2x}{dt^2} = -\kappa x - \sigma m\frac{dx}{dt} + q_-\bar{e}(t)\cdot\bar{i}_x, \tag{5.12}$$

where m is the mass of the electron. σ is the damping constant, q_- ($= -1.6 \times 10^{-19}$C) is the charge of the electron, and the applied field $\bar{e}(t)$ at $\bar{r} = 0$ (the location of the electron) is that due to a monochromatic electromagnetic wave at frequency ν, i.e.,

$$\bar{e}(t) = \bar{i}_x\, e(t) = \bar{i}_x\, \mathrm{Re}[E(\nu)\exp(-j2\pi\nu t)]. \tag{5.13}$$

In Eq. (5.12) on the righthand side. the first term is due to the restoring force of the spring, the second term is the damping force, and the last term is the electrodynamic force on the electron. Equation (5.12) describes a driven, damped, harmonic oscillator and its solution must also be monochromatic for a monochromatic driving field $e(t)$. Thus, the electron position $x(t)$ can be written as

$$x(t) = \mathrm{Re}[X(\nu)\exp(-j2\pi\nu t)], \tag{5.14}$$

where $X(\nu)$ is found by substituting (5.14) and (5.13) in (5.12), giving $-m\omega^2 X(\nu) = -\kappa X(\nu) + j\omega\sigma m X(\nu) + q_- E(\nu)$, or

$$X(\nu) = -\frac{(q_-/m)E(\nu)}{(\omega^2 - \omega_0^2) + j\sigma\omega}. \tag{5.15}$$

Here $\omega = 2\pi\nu$ and the natural frequency of oscillation of the bound electron is given by $\omega_0 = \sqrt{\kappa/m}$. The induced polarization is also in the $\bar{i}_x$ direction and is given by

$$\bar{p}(t) = p(t)\bar{i}_x = \bar{i}_x \mathrm{Re}[P(\nu)\exp(-j2\pi\nu t)], \tag{5.16}$$

where $P(\nu) = Nq_- X(\nu)$ with N being the number of interacting electrons per unit volume in the medium, which when combined with Eqs. (5.15) and (5.4) give the following expression for the susceptibility of the atomic medium:

$$\chi(\nu) = \frac{Nq_-^2/m\epsilon_0}{(\omega_0^2 - \omega^2) - j\sigma\omega}. \tag{5.17}$$

Decomposing into real and imaginary parts, we get

$$\chi'(\nu) = \frac{Nq_-^2}{m\epsilon_0}\frac{(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2\sigma^2} \simeq \frac{Nq_-^2}{2m\epsilon_0\omega_0}\frac{(\omega_0 - \omega)}{(\omega_0 - \omega)^2 + (\sigma/2)^2}, \tag{5.18}$$

$$\chi'' = \frac{Nq_-^2}{m\epsilon_0}\frac{\omega\sigma}{(\omega_0^2 - \omega^2)^2 + \omega^2\sigma^2} \simeq \frac{Nq_-^2}{2m\epsilon_0\omega_0}\frac{\sigma/2}{(\omega_0 - \omega)^2 + (\sigma/2)^2}, \tag{5.19}$$

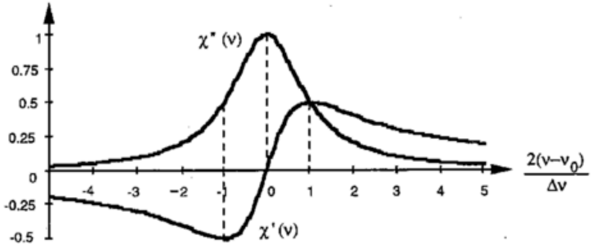

Figure 5.2: *Normalized absorption and dispersion curves.*

where the approximations are valid near resonance, i.e., $\omega \simeq \omega_0$. According to Eqs. (5.11) and (5.9), $\chi'(\nu)$ gives the refractive index of the medium and thus characterizes the dispersive response of the medium, whereas $\chi''(\nu)$ gives the attenuation or amplification constant and thus describes the absorptive or amplifying response of the medium. From Eq. (5.19), $\chi''(\nu)$ is always greater than zero giving an $\alpha(\nu)$ which is always less than zero implying that *in ordinary media light is never amplified*! Both $\chi'(\nu)$ and $\chi''(\nu)$ are plotted in Fig. 5.2. At $\nu = \nu_0$, $\chi'(\nu_0) = 0$ and $\chi''(\nu_0)$ is maximum. The full width at half maximum (FWHM) of $\chi''(\nu)$ is given by $\Delta\nu = \sigma/2\pi$ or $\Delta\omega = 2\pi\Delta\nu = \sigma$. The extrema of $\chi'(\nu)$ occur at $\nu_0 \pm \sigma/4\pi$. Far away from resonance, both the absorptive and dispersive responses vanish with the latter decaying rather slowly.

5.4 Normalized Line Shape Function

Using (2.14), the magnetic field associated with the electromagnetic wave whose electric field is given by (5.6) is obtained to be

$$\bar{H}(\bar{r}) = \frac{1}{j\mu_0\omega}\nabla \times \bar{E}(\bar{r}) = \frac{\alpha + j\beta}{j\mu_0\omega} E_0\bar{i}_y \exp[(\alpha + j\beta)z]. \tag{5.20}$$

Substituting (5.6) and (5.20) in (2.19), the intensity of the wave propagating along the z direction is given by

$$\begin{aligned} I_\nu(z) &= \left|\mathrm{Re}[\frac{1}{2}\bar{E}(\bar{r}) \times \bar{H}^*(\bar{r})]\right| \\ &= \frac{\beta}{2\mu_0\omega} E_0^2 \exp(2\alpha z) \\ &\equiv I_\nu(0)\exp(az), \end{aligned} \tag{5.21}$$

where $I_\nu(0) \equiv E_0^2\beta/2\mu_0\omega$. Thus the intensity attenuation (or amplification) coefficient a is twice the field attenuation (or amplification) coefficient α. Using (5.9), (5.11), and the expression (5.19) for $\chi''(\nu)$ obtained via the electron oscillator model, we get

$$\begin{aligned} a(\nu) &= \frac{-2k^2}{2nk}\frac{Nq_-^2}{2m\epsilon_0\omega_0}\frac{\sigma/2}{(\omega_0-\omega)^2+(\sigma/2)^2} \\ &= -\frac{Nq_-^2 k}{2nm\epsilon_0\omega_0}\frac{1}{2\pi}\frac{\Delta\nu/2}{(\nu_0-\nu)^2+(\Delta\nu/2)^2} \\ &\equiv -\frac{Nq_-^2 k}{2nm\epsilon_0\omega_0}\frac{1}{2}g(\nu) \simeq -\frac{Nq_-^2}{4nmc\epsilon_0}g(\nu), \end{aligned} \tag{5.22}$$

where $g(\nu)$ is the *normalized line-shape function* defined by

$$g(\nu) \equiv \frac{\Delta\nu/2\pi}{(\nu-\nu_0)^2+(\Delta\nu/2)^2} = \frac{\Delta\omega}{(\omega-\omega_0)^2+(\Delta\omega/2)^2}. \tag{5.23}$$

It is easily verified that $\int_{-\infty}^{\infty} g(\nu)d\nu = 1$. Also from Eqs. (5.18) and (5.19), we see that

$$\frac{\chi''(\nu)}{\chi'(\nu)} = \frac{\sigma}{2(\omega_0-\omega)} = \frac{\Delta\nu}{2(\nu_0-\nu)}, \tag{5.24}$$

which seems *independent of the detailed model* leading to (5.18) and (5.19). It turns out that equation (5.24) holds quite generally.

In the next lecture, we will explore ways by which a positive value for $a(\nu)$ can be achieved.

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Lecture 6 – EECS 379

ENERGY LEVELS AND RATE EQUATIONS

Reading Assignment: YARIV – Secs. 5.1, 5.2, 5.3, and 5.5.

6.1 Introduction

In the previous lecture, we saw that a classical electron-oscillator model of an atom gives a negative amplification coefficient for a wave propagating in the medium. To see how a positive coefficient is obtained, we need to consider the quantum mechanical properties of the atoms constituting the medium. We will do so, however, in a manner that relies minimally on the quantum mechanical concepts.

6.2 Atomic Energy Levels

According to quantum mechanics, the electrons orbiting the nuclei in the atoms of a medium occupy discrete energy levels, a few of which are shown in Fig. 6.1. Under the action of an applied electromagnetic field, the electrons make the so called "quantum jumps" between the various energy levels, emitting or absorbing discrete quanta of electromagnetic energy called photons. For example, if a monochromatic wave of frequency ν is incident on an atom

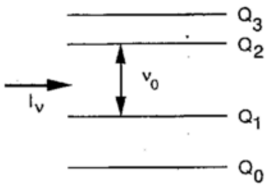

Figure 6.1: *Typical energy-level diagram of an atom.*

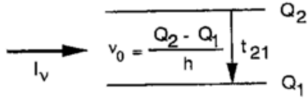

Figure 6.2: *Energy-level diagram of a two-level atom.*

whose energy levels are shown in Fig. 6.1, then an electron in level 2 can make a transition to level 1 emitting a photon of frequency $\nu \simeq \nu_0 \equiv (Q_2 - Q_1)/h$ in the process. The emitted photon is *coherent* with the incident photons, i.e., the spatio-temporal dependence of the electromagnetic field associated with the emitted photon is identical to that associated with the incident photons. Here h is Planck's constant ($= 6.67 \times 10^{-34}$ Js), and Q_i is the electron energy in the ith level. Vice versa, an atom in level 1 can make a transition to level 2, if $\nu \simeq \nu_0$, absorbing a photon of frequency ν in the process. Furthermore, an electron in level 2 can jump down to level 1 (or 0), spontaneously emitting a photon of frequency ν_0 [or $(Q_2 - Q_0)/h$], even in the absence of an applied electromagnetic field. This process takes place because all electrons eventually like to be in the ground state of their respective atoms; ground state being the state of lowest possible energy. In this case, the emitted photon is *incoherent* because the electromagnetic field associated with it has no relationship with the applied field. Transitions of the former kind (which depend upon the presence of an applied electromagnetic field) are called *induced* or *stimulated transitions* whereas those of the latter kind are called *spontaneous transitions.*

6.3 Spontaneous and Stimulated Transition Rates

Let N be the number of atoms per unit volume in a medium whose atoms have only two relevant energy levels as shown in Fig. 6.2. The electrons of the atoms can then only be in these two energy levels. Let N_1 (N_2) be the number of atoms per unit volume whose electrons are in level 1 (2), then

$$N = N_1 + N_2. \tag{6.1}$$

Furthermore, let I_ν be the intensity of a monochromatic plane wave of frequency ν propagating in this medium along the z direction. Then, quantum mechanics provides the following transition rates for the electrons of these atoms.

a) **Absorption**: The rate at which photons of the incident light are absorbed, causing the electrons of the atoms to make transitions from levels 1 to 2 in the process, is given by

$$\left.\frac{dN_2}{dt}\right|_{1\to 2} = W_i(\nu)N_1, \tag{6.2}$$

where

$$\begin{aligned} W_i(\nu) &= \frac{\lambda^2 I_\nu}{8\pi h\nu\, t_{21}} g(\nu) \\ &\equiv B_{12} I_\nu / c \end{aligned} \tag{6.3}$$

is the induced transition rate to be interpreted later, and B_{12} is the Einstein B coefficient. If ρ_ν is the density of photons of frequency ν in the medium, then

$$I_\nu = h\nu c \rho_\nu, \tag{6.4}$$

and the induced transition rate can also be written as

$$W_i(\nu) = h\nu B_{12}\rho_\nu. \tag{6.5}$$

b) **Stimulated Emission**: The rate at which photons are coherently added to the incident light field and the electrons of the atoms make transitions from levels 2 to 1 in the process, is given by

$$\left.\frac{dN_2}{dt}\right|_{2\to 1}^{\mathrm{st}} = -W_i(\nu)N_2. \tag{6.6}$$

c) **Spontaneous Emission**: The rate at which the atoms make spontaneous transitions from levels 2 to 1, emitting incoherent photons in the process, is given by

$$\begin{aligned} \left.\frac{dN_2}{dt}\right|_{2\to 1}^{\mathrm{sp}} &= -\frac{N_2}{t_{21}} \\ &= -A_{21}N_2, \end{aligned} \tag{6.7}$$

where $A_{21} = 1/t_{21}$ is the Einstein A coefficient or the spontaneous decay rate; it is a characteristic of the atoms. Note that, as intuitively expected, the rate of absorption is proportional to N_1, the number of atoms per unit volume in the lower state of the atoms [cf. Eq. (6.2)]. Similarly, the rate of stimulated or spontaneous emission is proportional to N_2 [cf. Eqs. (6.6) or (6.7)]. Furthermore, the absorption and stimulated emission rates are proportional to I_ν via the induced transition rate $W_i(\nu)$ of Eq. (6.3), which can be simply interpreted in the following way: $\lambda^2 I_\nu/h\nu$ is the number of photons arriving per second in an area λ^2 in the vicinity of an atom and $g(\nu)/t_{21}$ is of order 1 for $\nu \simeq \nu_0$. Here, $g(\nu)$ is the normalized line shape function defined via Eq. (5.23) in the previous lecture. For $\nu \simeq \nu_0$, $g(\nu) \simeq 2/\pi\Delta\nu$; therefore, $g(\nu)/t_{21} \simeq 2/\pi\Delta\nu t_{21}$. But, $\Delta\nu = \sigma/2\pi$, i.e., it is proportional to the damping rate in the electron-oscillator model considered in the previous lecture. According to quantum theory, however, an excited atom in level 2 is damped via spontaneous emission with rate $1/t_{21}$. Thus, the width of the normalized line shape function provided by the quantum theory is expected to be $\Delta\nu \simeq t_{21}^{-1}$ showing that $\Delta\nu t_{21} \simeq 1$ or $g(\nu)/t_{21} \simeq 2/\pi$. Thus, it follows that the rate of stimulated emission is proportional to the rate of arrival of the incident photons in an area λ^2 in the vicinity of the atom; a result which is intuitively very clear.

6.4 Rate Equations

Equations (6.2), (6.6) and (6.7) are the building blocks of more complicated rate equations involving many more levels of the atoms. Corresponding to these atomic rate equations, there is a rate equation for the growth or decay of the incident wave. Noting that with each atomic up or down transition, there is a corresponding absorbed or emitted photon from or to the incident wave, we have

$$\frac{d\rho_\nu}{dt} = -\left.\frac{dN_2}{dt}\right|_{\text{total}} = -\left(\left.\frac{dN_2}{dt}\right|_{1\to 2} + \left.\frac{dN_2}{dt}\right|_{2\to 1}^{\text{st}} + \left.\frac{dN_2}{dt}\right|_{2\to 1}^{\text{sp}}\right). \tag{6.8}$$

Since $\dfrac{d\rho_\nu}{dt} = c\dfrac{d\rho_\nu}{dz}$, using Eqs. (6.4), the following rate equation is easily obtained:

$$\frac{dI_\nu}{dz} = -h\nu\left(\left.\frac{dN_2}{dt}\right|_{1\to 2} + \left.\frac{dN_2}{dt}\right|_{2\to 1}^{\text{st}}\right), \tag{6.9}$$

where we have neglected the spontaneous emission term. This is justifiable whenever the incident intensity is large. Substituting Eqs. (6.2) and (6.6), the above equation becomes

$$\begin{aligned}\frac{dI_\nu}{dz} &= -h\nu\left[W_i(\nu)N_1 - W_i(\nu)N_2\right] \\ &= h\nu\, W_i(\nu)(N_2 - N_1). \end{aligned} \tag{6.10}$$

Note that in Eq. (6.9), we have not included the spontaneously emitted photons because they *do not add coherently* to the incident wave. In fact, these photons degrade the monochromatic nature of the incident wave. For a large I_ν, the contribution of these photons is negligible, i.e., the induced transition rate is much larger than the spontaneous transition rate [cf. Eqs. (6.2)-(6.7)]. Substituting (6.2) in (6.10), we get

$$\frac{dI_\nu}{dz} = \frac{\lambda^2 g(\nu)(N_2 - N_1)}{8\pi t_{21}} I_\nu \tag{6.11}$$

which when compared with the derivative of Eq. (5.21) gives

$$a(\nu) = \frac{\lambda^2\, g(\nu)(N_2 - N_1)}{8\pi\, t_{21}}. \tag{6.12}$$

Thus, we have a mechanism for gain: if somehow $N_2 > N_1$, then $a(\nu) > 0$ and the intensity of the incident wave will grow. At temperature T, however, thermodynamics dictates that

$$\frac{N_2}{N_1} = \exp[-(Q_2 - Q_1)/k_B t], \tag{6.13}$$

where k_B is the Boltzman constant. At T = 300K (room temperature) for $\nu_0 \simeq 5 \times 10^{14}$Hz, this gives $N_2/N_1 = \exp(-h\nu_0/k_B t) \simeq \exp(-80) \simeq 0$. Therefore, under normal conditions $N_2 \ll N_1$ leading to a negative $a(\nu)$, consistent with the classical electron-oscillator model. In the next lecture, we will see how $N_2 > N_1$ can be achieved. We conclude this lecture by filling in a few details.

Noting that $a(\nu) = 2\alpha(\nu)$ and using Eqs. (5.9) and (5.11), we get

$$\chi''(\nu) = -\frac{n}{k}a(\nu) = -\frac{n\lambda^3 g(\nu)(N_2 - N_1)}{16\pi^2 t_{21}}. \tag{6.14}$$

$\chi'(\nu)$ is obtained using Eq. (5.24), which we have already noted to be independent of any particular model for the gain.

When I_ν is zero and (N_2/N_1) is not given by (6.13), but let us say $N_2 > N_1$, then $(N_2 - N_1)$ decays to the equilibrium value given by (6.13) via Eq. (6.7) whose solution is

$$N_2(t) = N_2(0)\exp(-t/t_{21}). \tag{6.15}$$

$N_2(t) - N_1(t)$ can be calculated using Eq. (6.1).

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Lecture 7 – EECS 379

POPULATION INVERSION AND GAIN SATURATION

Reading Assignment: YARIV – Secs. 5.5 and 5.6.

7.1 Population Inversion

We noted in the previous lecture that in order to obtain a positive amplification coefficient $a(\nu)$ [cf. Eq. (6.12)], one needs $N_2 > N_1$. However, in thermal equilibrium, Eq. (6.13) dictates that $N_2 \ll N_1$. Thus, in order to obtain gain we need to somehow break the thermal equilibrium and create the so called *population inversion* in the amplifying medium. It turns out that if we consider only two energy levels of an atom, it is imposible to achieve the $N_2 > N_1$ condition. As we will see in a homework problem, the best one can achieve is $N_2 = N_1$ or $N_2 - N_1 = 0$ implying zero absorption even for a normally absorptive medium. This phenomenon is called *absorption saturation* or *bleaching*.

7.2 Population Dynamics in a Four-Level System

Consider a medium consisting of N identical atoms per unit volume whose relevant energy levels are shown in Fig 7.1 along with the various decay rates. The condition $N_2 > N_1$ can also be achieved using only three energy levels of an atom and will be considered in another homework problem. If N_i is the density of atoms in level i, then we must have

$$N = N_0 + N_1 + N_2 + N_3. \tag{7.1}$$

The decay rates $t_3^{-1}, t_{31}^{-1}, t_{32}^{-1}, t_2^{-1}, t_{21}^{-1}, t_1^{-1}$, respectively, are the rates with which the electrons in the respective upper level decay to the corresponding lower level in a free atom. For example, t_3^{-1} is the rate with which the electrons in level 3 decay to level 0 via spontaneous emission or any other non-radiative process. These rates are properties of a specific atom

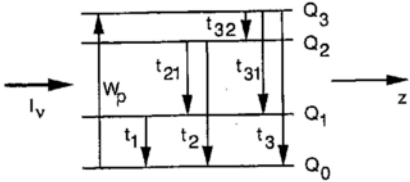

Figure 7.1: *Energy-level diagram of a four-level atom.*

and can either be theoretically calculated or experimentally measured. They can also be modified by influencing the surroundings of the atom. Further assuming that only t_{32}^{-1}, t_{21}^{-1}, and t_1^{-1} are nonzero, i.e., $t_{31}^{-1} = t_3^{-1} = t_2^{-1} = 0$ (in essence, we are assuming that the electrons of the atoms, for example, once in level 3 can not jump directly to level 1 etc.), we obtain the following rate equations for the population densities of the various levels:

$$\frac{dN_3}{dt} = W_p N_0 - \frac{N_3}{t_{32}}, \tag{7.2}$$

$$\frac{dN_2}{dt} = \frac{N_3}{t_{32}} - \frac{N_2}{t_{21}}, \tag{7.3}$$

$$\frac{dN_1}{dt} = \frac{N_2}{t_{21}} - \frac{N_1}{t_1}, \tag{7.4}$$

$$\frac{dN_0}{dt} = \frac{N_1}{t_1} - W_p N_0. \tag{7.5}$$

Here, W_p is the pumping rate with which the electrons are transferred from level 0 to 3. The pumping process is responsible for breaking the thermal equilibrium and is realized in practice in many ways. For example, in gas lasers such as helium-neon or argon-ion (neon is the active atom in the former and singly ionized argon in the latter), the energetic electrons of an electric discharge collide with the active atoms to push the atomic electrons to the relevant upper levels.

Under steady state pumping conditions, the rate Eqs. (7.2)-(7.5) can be easily solved by setting the time derivatives equal to zero. Equations (7.2) and (7.5) give, respectively,

$$N_3^0 = W_p t_{32} N_0^0, \tag{7.6}$$

$$N_1^0 = W_p t_1 N_0^0, \tag{7.7}$$

whereas (7.3) combined with (7.2) leads to

$$N_2^0 = N_3^0 \frac{t_{21}}{t_{32}} = W_p t_{21} N_0^0. \tag{7.8}$$

Substituting Eqs. (7.6)-(7.8) in (7.1), we get

$$N_0^0 = \frac{N}{1 + W_p(t_1 + t_{32} + t_{21})}. \tag{7.9}$$

In practice, one chooses the levels such that $t_{21} \gg t_1, t_{32}$, simplifying the above equation to

$$N_0^0 \simeq \frac{N}{1 + W_p t_{21}}, \tag{7.10}$$

which when combined with Eqs. (7.6)-(7.8) yields

$$N_1^0 \simeq \frac{W_p t_1 N}{1 + W_p t_{21}} \simeq 0, \tag{7.11}$$

$$N_2^0 \simeq \frac{W_p t_{21} N}{1 + W_p t_{21}}, \tag{7.12}$$

$$N_3^0 \simeq \frac{W_p t_{32} N}{1 + W_p t_{21}} \simeq 0. \tag{7.13}$$

Thus, in steady state, the population inversion is given by

$$N_2^0 - N_1^0 \simeq \frac{W_p N}{1 + W_p t_{21}}(t_{21} - t_1) \simeq \frac{W_p N t_{21}}{1 + W_p t_{21}}, \tag{7.14}$$

which is greater than zero for any $W_p \neq 0$. The above equation is further simplified to

$$\Delta N^0 \equiv N_2^0 - N_1^0 = W_p N t_{21}, \tag{7.15}$$

because for many laser systems $W_p t_{21} \ll 1$. Finally, substituting Eq. (7.15) into (6.12), we get the following amplification coefficient:

$$a_0(\nu) = \frac{\lambda^2 N W_p}{8\pi} g(\nu), \tag{7.16}$$

where we have subscripted a with a 0 for reasons which will become clear in the next section.

7.3 Gain Saturation

According to Eqs. (7.16) and (6.11), the incident light intensity I_ν is amplified exponentially,

$$I_\nu(z) = I_\nu(0)\exp[a_0(\nu)z], \tag{7.17}$$

even under steady state conditions. This would imply that for a nonzero W_p, the input intensity would continue to grow without bound as the beam propagates in the medium. Such an unphysical result is due to the fact that in the rate equations (7.2)-(7.5), we have neglected to include the stimulated transition terms discussed in Sec. 6.3 [Eqs. (6.2) and (6.6)]. These terms are proportional to I_ν and hence can only be neglected close to the input end of the amplifying medium. Once I_ν grows to a large value (how large? we will see later), these terms play a crucial role in determining the net gain coëfficient.

Because of the presence of $g(\nu)$ in the gain coefficient $a(\nu)$ [cf. Eqs. (7.16) and (5.23)], the latter is significant only for $\nu \simeq \nu_0$. Therefore, we assume that the input beam has a frequency $\nu \simeq \nu_0$. Note that $\Delta\nu$ appearing in $g(\nu)$ [see Eq. (5.23)] determines the bandwidth of the optical amplifier. Since. such a beam can cause transitions between levels 1 and 2 only, the rate equations (7.2) and (7.5) stay the same whereas (7.3) and (7.4) get modified to [cf. Eqs. (6.2) and (6.6)]

$$\begin{aligned}
\frac{dN_2}{dt} &= \frac{N_3}{t_{32}} - \frac{N_2}{t_{21}} + W_i(\nu)N_1 - W_i(\nu)N_2, && (7.18)\\
\frac{dN_1}{dt} &= \frac{N_2}{t_{21}} - \frac{N_1}{t_1} - W_i(\nu)N_1 + W_i(\nu)N_2. && (7.19)
\end{aligned}$$

Once again, in steady state, by setting the time derivatives in Eqs. (7.2), (7.5), (7.18), and (7.19) equal to zero, we get

$$\begin{aligned}
N_3^0 &= W_p N_0^0 t_{32}, && (7.20)\\
N_1^0 &= W_p N_0^0 t_1, && (7.21)\\
0 &= \frac{N_3^0}{t_{32}} - \frac{N_2^0}{t_{21}} + W_i(\nu)[N_1^0 - N_2^0], && (7.22)\\
0 &= \frac{N_2^0}{t_{21}} - \frac{N_1^0}{t_1} - W_i(\nu)[N_1^0 - N_2^0]. && (7.23)
\end{aligned}$$

The above set of linear equations can be trivally solved by making the same assumptions as those made in Sec. 2, namely, $t_{21} \gg t_1, t_3$, and $W_p t_{21} \ll 1$ (we also assume $W_i(\nu)t_1 \ll 1$), to give

$$N_3^0 \simeq 0, \tag{7.24}$$

$$N_1^0 \simeq 0, \tag{7.25}$$

$$N_2^0 \simeq \frac{W_p N_0^0 t_{21}}{1 + W_i(\nu)t_{21}}. \tag{7.26}$$

Substituting the above two equations in (7.1), we get the following equation for the *saturated population inversion*:

$$\Delta N_s^0 \simeq N_2^0 = \frac{W_p N t_{21}}{1 + W_p t_{21} + W_i(\nu)t_{21}} \simeq \frac{W_p N t_{21}}{1 + W_i(\nu)t_{21}}, \tag{7.27}$$

which when further substituted in Eq. (6.12) gives the following *saturated gain coefficient*:

$$a(\nu) = \frac{\lambda^2 N W_p g(\nu)}{8\pi[1 + W_i(\nu)t_{21}]} = \frac{a_0(\nu)}{1 + W_i(\nu)t_{21}}, \tag{7.28}$$

where $a_0(\nu)$ is the so called *unsaturated gain coefficient* as given by Eq. (7.16) for $I_\nu = 0$. Equation (7.28) can also be written as

$$a(\nu) = \frac{a_0(\nu)}{1 + I_\nu / I_s(\nu)} \tag{7.29}$$

where the *saturation intensity* $I_s(\nu)$ defines the intensity for which the net gain drops to $\frac{1}{2}$ the unsaturated gain value and is given by [using Eq. (6.3)]

$$I_s(\nu) \equiv \frac{8\pi h\nu}{\lambda^2 g(\nu)}. \tag{7.30}$$

Thus, the unsaturated gain coefficient $a_0(\nu)$ can be used in Eq. (7.17) as long as $I_\nu \ll I_s(\nu)$. But as soon as I_ν approaches the saturation intensity $I_s(\nu)$, the saturated coefficient $a(\nu)$ of Eq. (7.27) must be used. Physically, the reduction in gain is due to depletion in the population inversion.

A laser is formed by placing the gain medium considered in this lecture inside a Fabry-Perot resonator considered in lecture 3.3. In the next lecture, we will consider this problem and derive the conditions which must be satisfied for laser action to take place.

NORTHWESTERN UNIVERSITY
Department of Electrical Engineering and Computer Science

Lecture 8 - EECS 379

LASER OSCILLATION

Reading Assignment: YARIV - Sec. 5.7.

8.1 Fabry-Perot Laser

Consider a Fabry-Perot (FP) resonator formed by mirrors of reflectivity R (and transmissivity T) as shown in Fig. 8.1. We assume that the mirrors with cross-sectional area A are lossless so that $R + T = 1$. To provide gain, let us suppose that the interior of the FP-resonator cavity from $z = 0$ to $z = \ell$ is filled with a medium consisting of 4-level atoms of the kind considered in Secs. 7.2 and 7.3. We would like to calculate the field

$$\bar{E}(\bar{r}) = i_x E_{\text{out}} \exp(j\frac{2\pi\nu}{c}z), \quad z > L. \tag{8.1}$$

of the output plane wave in the region $z > L$ resulting from an incident plane wave of frequency $\nu \simeq \nu_0$ whose electric field is given by

$$\bar{E}(\bar{r}) = i_x E_{\text{in}} \exp(j\frac{2\pi\nu}{c}z), \quad z < 0. \tag{8.2}$$

We do so by summing the fields of the multiply reflected plane waves within the FP cavity (cf. Lecture 4). Due to presence of the gain, each partially reflected component is multiplied by an $\exp[(\alpha + j\beta)\ell]$ factor in propagation from $z = 0$ to $z = \ell$ [cf. Eq. (5.6)]. Here, both α and β are functions of ν. From Eq. (5.21) $\alpha(\nu) = a(\nu)/2$, and using (5.11), (5.23), and (6.11)

$$\begin{aligned}
\beta(\nu) &= k\left[1 + \chi'(\nu)\right]^{1/2} \\
&\simeq k\left[1 + \chi'(\nu)/2\right] \\
&= k + n\frac{\nu - \nu_0}{\Delta\nu}\, a(\nu).
\end{aligned} \tag{8.3}$$

Superimposing all the partially reflected waves at the output, we get

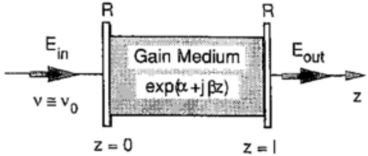

Figure 8.1: *Schematic of a generic Fabry-Perot laser.*

$$
\begin{aligned}
E_{\text{out}} &= E_{\text{in}}\Big\{\sqrt{T}\exp[(\alpha+j\beta)\ell]\sqrt{T} \\
&\quad + \sqrt{T}\exp[(\alpha+j\beta)\ell]\sqrt{R}\exp[(\alpha+j\beta)\ell]\sqrt{R}\exp[(\alpha+j\beta)\ell]\sqrt{T} \\
&\quad + \sqrt{T}R^2\exp[5(\alpha+j\beta)\ell]\sqrt{T}+\ldots\Big\} \\
&= \frac{T\exp[(\alpha+j\beta)\ell]}{1-R\exp[2(\alpha+j\beta)\ell]}E_{\text{in}}. \qquad (8.4)
\end{aligned}
$$

Since $I \propto |E|^2$, we obtain the following expression for the intensity gain

$$
\begin{aligned}
G(\nu) &\equiv \frac{I_{\text{out}}}{I_{\text{in}}} = |\frac{E_{\text{out}}}{E_{\text{in}}}|^2 \\
&= \frac{T^2\exp[a(\nu)\ell]}{|1-R\exp[\{a(\nu)+2j\beta(\nu)\}\ell]|^2} \qquad (8.5)
\end{aligned}
$$

8.2 Lasing Conditions

From Eq. (8.5), we see that in steady state there can be finite I_{out} even when $I_{\text{in}} = 0$. This happens when the denominator in the right member of Eq. (8.5) equals zero, i.e.,

$$R\exp[\{a(\nu)+2j\beta(\nu)\}\ell] = 1. \qquad (8.6)$$

The above is a complex equation implying the following two conditions for oscillation (laser action) to build up within the Fabry-Perot resonator:

$$
\begin{aligned}
a(\nu) &= -\frac{1}{\ell}\ln R, \qquad (8.7) \\
\beta(\nu) &= m\pi/\ell. \qquad (8.8)
\end{aligned}
$$

The former is a statement of energy conservation in steady state – the round-trip fractional gain $\exp[a(\nu)\ell]$ must compensate for the round-trip fractional loss R – whereas the latter is

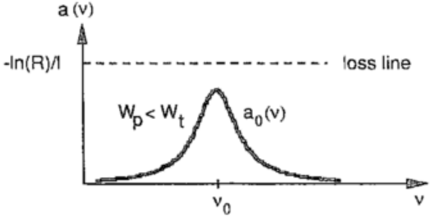

Figure 8.2: *Gain and loss for $W_p < W_t$.*

the Fabry-Perot resonance condition giving frequency values ν_m for which oscillation builds up. Substituting $\beta(\nu)$ from Eq. (8.3) into Eq. (8.8), we get

$$\nu + n\frac{\nu - \nu_0}{\Delta\nu}\,\frac{c}{2\pi}a(\nu_0) = \frac{mc}{2\ell}, \tag{8.9}$$

where, to a very good approximation, we have put $\nu = \nu_0$ in $a(\nu)$. The above equation when solved assuming $n\,c\,a(\nu_0)/2\pi\Delta\nu \ll 1$ gives

$$\nu_m = \frac{mc}{2\ell}. \tag{8.10}$$

The shift in ν_m, when $nc\,a(\nu_0)/2\pi\Delta\nu$ is not negligible, is approximately given by $n\,c\,a(\nu_0)(\nu_0-\nu_m)/2\pi\Delta\nu$ and results in what is known as the *frequency pulling effect.*

8.3 Lasing Threshold

The first condition, Eq. (8.7), determines the minimum pumping rate W_p needed for laser action to start and the resulting output power $P_{\rm out}$. Using Eq. (7.29), this condition can be written as

$$\frac{a_0(\nu)}{1 + I_\nu/I_s(\nu)} = -\frac{1}{\ell}\,\ln R. \tag{8.11}$$

When $W_p = 0$, from Eq. (7.16), $a_0(\nu)$ is also zero and the above equality is not satisfied. No laser action takes place and I_ν, the intensity within the FP resonator, stays zero. (Remember, we are assuming that $I_{\rm in} = 0$.) For $W_p > 0$, $a_0(\nu) > 0$ but I_ν remains zero until the

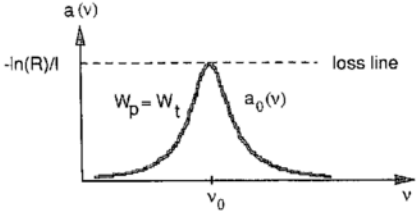

Figure 8.3: *Gain and loss for $W_p = W_t$.*

above equality is satisfied, i.e., until $a_0(\nu) = -(1/\ell)\ln R$. Equation (7.16) then determines a minimum $W_p \equiv W_t$, called the *threshold pumping rate*, given by

$$W_t = -\frac{8\pi \ln R}{\lambda^2 \ell N g(\nu)} \tag{8.12}$$

at which the round-trip fractional gain $\exp[a(\nu)\ell]$ exactly compensates for the round-trip fractional loss R. For $W_p < W_t$, $a_0(\nu) < -(1/\ell)\ln R$ resulting in the round-trip gain that is less than the round-trip loss. Figure 8.2 illustrates the $W_p < W_t$ case and $W_p = W_t$ case is shown in Fig. 8.3. In the former case, not only does I_ν stay zero, any initial I_ν will also decay to zero. Whereas in the latter, any initial I_ν will maintain its value because the round-trip fractional gain equals the round-trip fractional loss.

8.4 Output Power

When $W_p > W_t$, the unsaturated gain coefficient $a_0(\nu)$ exceeds the loss coefficient $(-1/\ell)\ln R$ over a range of frequencies as illustrated in Fig. 8.4. The round-trip fractional gain is larger than the round-trip fractional loss and light at these frequencies is amplified as it bounces back and forth between the two mirrors of the FP resonator. However, positive feedback or constructive interference occurs only if there happens to be a cavity resonance ν_m at some frequency in the band that is amplified. We suppose that the FP-cavity length is such that $\nu_m = mc/2\ell = \nu_0$, that is, the approximate cavity resonance condition (8.10) is satisfied at the maximum of the gain curve. In this case, laser action will occur at $\nu = \nu_0$ and I_{ν_0} will

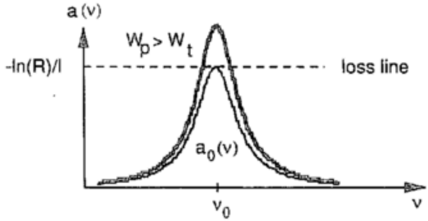

Figure 8.4: *Gain and loss for $W_p > W_t$.*

grow even when $I_{\text{in}}(\nu_0) = 0$.

Because at $\nu = \nu_0$, the unsaturated gain coefficient is larger than the loss coefficient, it would seem that I_{ν_0} will grow without a bound. However, as I_{ν_0} increases from a negligible value, the net gain (gain minus the loss) is determined by the *saturated gain coefficient* $a(\nu_0)$ and not by the unsaturated gain coefficient $a_0(\nu_0)$. I_{ν_0} will then grow to a steady-state value at which the saturated round-trip fractional gain equals the round-trip fractional loss as illustrated by the thin curve in Fig. 8.4. From (8.11), the steady-state I_{ν_0} is governed by

$$a(\nu_0) = \frac{a_0(\nu_0)}{1 + I_{\nu_0}/I_s(\nu_0)} = -\frac{1}{\ell}\ln R,$$

which when solved for I_{ν_0} gives

$$I_{\nu_0} = I_s(\nu_0).\left[-\frac{a_0(\nu_0)\ell}{\ln R} - 1\right]. \tag{8.13}$$

Here $I_s(\nu_0)$ is the saturation intensity at ν_0 [cf. Eq. (7.30)]. Using Eqs. (7.16) and (8.12),

$$I_{\nu_0} = \left(\frac{W_p}{W_t} - 1\right) I_s(\nu_0), \tag{8.14}$$

which shows that above threshold the steady-state intensity is linearly proportional to the pumping rate. I_{ν_0}, however, is the intensity within the FP resonator. The output power P_{out} from one end of the resonator can be calculated by multiplying $I_{\nu_0}/2$ with the transmissivity T and area A of the end mirror, giving

$$P_{\text{out}} = TA\frac{I_s(\nu_0)}{2}\left(\frac{W_p}{W_t} - 1\right)$$

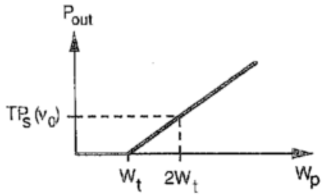

Figure 8.5: *Output power versus the pumping rate for a generic laser.*

$$= \quad T\frac{P_s(\nu_0)}{2}\left(\frac{W_p}{W_t}-1\right), \tag{8.15}$$

where $P_s(\nu_0)$ represents the saturation power of the laser at ν_0. The $1/2$ factor arises because the total intensity at any point within the resonator has equal contributions from a left-going wave and a right-going wave. In Fig. 8.5, we have plotted the output power $P_{\rm out}$ as a function of the pumping rate W_p. The slope above threshold is given by $TP_s(\nu_0)/W_t$ which is inversely proportional to W_t. This slope directly determines the efficiency with which power from the pumping source is converted into light power.